

Abdus Salam School of Mathematical Sciences-GCU Lahore
Ph.D. Admission Test - April 2009

Problem 1. Show that in any triangle $\triangle ABC$ the following inequalities hold:

$$\cos A + \cos B + \cos C \leq \frac{3}{2}. \text{ What is the minimal value of the sum?}$$

Problem 2. A 3×3 real matrix is called *magic* if there is a real number S such that in each row, column, and diagonal the sum of the elements equals S .

a) Show that the set of magic matrices is a real vector space and find its dimension.

b) Can you find a basis of this vector space with integral matrices?

Problem 3. A continuous function $f : D \rightarrow \mathbb{R}$ is defined on the set $D = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$.

a) Show that for any $r \in [1, 2]$ there is an element $z_r \in D$, $|z_r| = r$ such that $f(z_r) = \sup_{|z|=r} f(z) = M_r$; also, for any $\alpha \in [0, 2\pi]$ there is an element $z_\alpha \in D$, $\arg z_\alpha = \alpha$, such that $f(z_\alpha) = \inf_{\arg z=\alpha} f(z) = A_\alpha$.

b) Put in increasing order the two numbers:

$$M = \inf_{r \in [1, 2]} M_r \text{ and } A = \sup_{\alpha \in [0, 2\pi]} A_\alpha .$$

Problem 4. a) Show that any finitely generated subgroup of the group $(\mathbb{Q}, +)$ is cyclic, but the group itself is not cyclic.

b) Can you find a surjective function $f : \mathbb{N} \rightarrow \mathbb{Q} \setminus \{0\}$ such that:

$$f(xy) = f(x)f(y) ?$$

Problem 5. For a continuous function $f : [0, 1] \rightarrow \mathbb{R}$, compute

$$\lim_{n \rightarrow \infty} \int_0^1 n x^n f(x) dx \text{ (particular case : } f(x) \text{ is a polynomial function).}$$

Time: 3 hours.