The Darboux criterion of integrability

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If $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable, then f is bounded.

Problem: Characterize integrability within the class of bounded functions.

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. Given a division $\Delta = (x_k)_{k=0}^n$ of $[a, b]$ we attach to it the bounds

$$\begin{aligned} m_k(\Delta) &= \min_{x \in [x_k, x_{k+1}]} f(x) \\ M_k(\Delta) &= \max_{x \in [x_k, x_{k+1}]} f(x). \end{aligned}$$

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*Lecture presented on December 17, 2008, at the Abdus Salam School of Mathematical Sciences, Lahore.

and the *lower/upper Darboux* integral sums:

$$\begin{aligned} LDS_{\Delta}(f) &= \sum_{k=0}^{n-1} m_k(\Delta)(x_{k+1}-x_k) \\ UDS_{\Delta}(f) &= \sum_{k=0}^{n-1} M_k(\Delta)(x_{k+1}-x_k). \end{aligned}$$

Clearly

$$LDS_{\Delta}(f) \le RS_{(\Delta,\boldsymbol{\xi})}(f) \le UDS_{\Delta}(f)$$

for any possible choice of the intermediate Δ - point system and it is also clear that

$$\begin{aligned} LDS_{\Delta}(f) &= \inf_{\boldsymbol{\xi}} RS_{(\Delta,\boldsymbol{\xi})}(f) \quad \text{and} \\ UDS_{\Delta}(f) &= \sup_{\boldsymbol{\xi}} RS_{(\Delta,\boldsymbol{\xi})}(f). \end{aligned}$$

Lemma. *If* Δ_2 *is finer than* Δ_1 *then*

$$LDS_{\Delta_1}(f) \le LDS_{\Delta_2}(f) \le UDS_{\Delta_2}(f) \le LDS_{\Delta_1}(f).$$

Corollary. *For any two divisions Δ_1 and Δ_2,*

$$LDS_{\Delta_1}(f) \le UDS_{\Delta_2}(f).$$

that is, every lower Darboux sum does not exceed any upper Darboux sum.

Proof. In fact,

$$\begin{aligned} LDS_{\Delta_1}(f) &\le LDS_{\Delta_1 \cup \Delta_2}(f) \le UDS_{\Delta_1 \cup \Delta_2}(f) \\ &\le UDS_{\Delta_2}(f). \quad \blacksquare \end{aligned}$$

This situation leads us to the definition of

$$\underline{I}(f) = \sup_{\Delta} LDS_{\Delta}(f),$$

called *the lower Darboux integral* of f and

$$\overline{I}(f) = \inf_{\Delta} UDS_{\Delta}(f),$$

called *the upper Darboux integral* of f. Obviously,

$$\underline{I} \le \overline{I}.$$

Theorem (Darboux Criterion of Riemann Integrability). *A function* $f : [a, b] \to \mathbb{R}$ *is Riemann integrable if and only if for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$UDS_{\Delta}(f) - LDS_{\Delta}(f) < \varepsilon$$

for all divisions Δ *with* $||\Delta|| < \delta$.

Proof. We will show first that the condition is necessary. Let $I = \int_a^b f(x)dx$ and let $\varepsilon > 0$. By the definition of integrability, there is $\delta > 0$ such that

$$I - \frac{\varepsilon}{2} < RS_{(\Delta,\boldsymbol{\xi})}(f) < I + \frac{\varepsilon}{2}$$

for every division Δ with $||\Delta|| < \delta$ and every intermediate Δ - point system. Thus, from the properties of **inf** and **sup**,

$$I - \frac{\varepsilon}{2} < UDS_{\Delta}(f) \leq I + \frac{\varepsilon}{2}$$

and

$$I - \frac{\varepsilon}{2} \leq LDS_{\Delta}(f) < I + \frac{\varepsilon}{2}.$$

This implies that

$$UDS_{\Delta}(f) - LDS_{\Delta}(f) < (I + \frac{\varepsilon}{2}) - (I - \frac{\varepsilon}{2}) = \varepsilon$$

for every division Δ with $||\Delta|| < \delta$. We will show now that the condition is sufficient. Since

$$0 \le \overline{I}(f) - \underline{I}(f) \le UDS_{\Delta}(f) - LDS_{\Delta}(f) < \varepsilon$$

for every $\varepsilon > 0$, we get that $\overline{I}(f) = \underline{I}(f)$. Let I be the common value of the two Darboux integrals. We will show that f is Riemann integrable and that $\int_a^b f(x)dx = I$. We know that

$$\begin{aligned} LDS_{\Delta}(f) &\le I \le UDS_{\Delta}(f) \quad \text{and} \\ LDS_{\Delta}(f) &\le RS_{(\Delta,\boldsymbol{\xi})}(f) \le UDS_{\Delta}(f). \end{aligned}$$

Then

$$|I - RS_{(\Delta,\boldsymbol{\xi})}(f)| \le UDS_{\Delta}(f) - LDS_{\Delta}(f) < \varepsilon$$

when $||\Delta|| < \delta$. This completes the proof. ■

Consequences of the Darboux Criterion:

Theorem (Inheritance of integrability). *If f is a Riemann integrable function on $[a, b]$ and $[c, d] \subset [a, b]$, then f is Riemann integrable on $[c, d]$.*

Theorem (Additivity of integral). *If f is Riemann integrable on $[a, c]$ and on $[c, b]$, then f is Riemann integrable on $[a, b]$ and*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

A set $X \subset \mathbb{R}$ is called a *Jordan null set* if for any $\varepsilon > 0$ there is a finite family of compact intervals $([a_k, b_k])_{k=1}^n$ such that $X \subset \cup_{k=1}^n [a_k, b_k]$ and $\sum_{k=1}^n (b_k - a_k) < \varepsilon$. In other words, if X can be covered by a finite number of intervals of arbitrary small total length. If in the previous definition instead of compact intervals we use bounded open intervals we define the same concept. Indeed, if the property is true for a finite family of bounded open intervals then it will also be true for their closures. Conversely, let $([a_k, b_k])_{k=1}^n$ be a family of compact intervals such that $X \subset \cup_{k=1}^{n}[a_k, b_k]$ and $\sum_{k=1}^{n}(b_k - a_k) = L < \varepsilon$. Let

$$a_k' = a_k - \frac{\varepsilon - L}{3n} \quad and \quad b_k' = b_k + \frac{\varepsilon - L}{3n}.$$

Then $X \subset \bigcup_{k=1}^{n}(a_k', b_k')$ and

$$\begin{aligned} \sum_{k=1}^{n}(b_k' - a_k') &= \sum_{k=1}^{n}(b_k - a_k) + \frac{2(\varepsilon - L)}{3} \\ &= L + \frac{2(\varepsilon - L)}{3} < \varepsilon. \end{aligned}$$

Properties of Jordan null sets: a Jordan null set is bounded. The closure of a Jordan null set is a compact Jordan null set. Any finite set is a Jordan null set. We will give a proof of this last statement. Let $X = \{a_1, a_2, \ldots, a_n\}$ and $\varepsilon > 0$. Then $([a_k - \frac{\varepsilon}{3n}, a_k + \frac{\varepsilon}{3n}])_{k=1}^{n}$ is a cover of X of total length $< \varepsilon$.

Theorem (Property of stability). *Let* $f, g : [a, b] \to \mathbb{R}$ *and* X *a Jordan null subset of* $[a, b]$ *such that* $f(x) =$

$g(x)$ for all $x \in [a,b] \setminus X$. If f is Riemann integrable then g is Riemann integrable and $\int_a^b f(x)dx = \int_a^b g(x)dx$.

In other words, changes on Jordan null sets influence neither the character of integrability nor the value of the integral.

Proof. Notice first that if $\Delta = (x_k)_{k=0}^n$ is a division of $[a,b]$ and $([x_k, x_{k+1}])_{k=p}^q$ are the subintervals of Δ intersecting an interval $[c,d] \subset [a,b]$, then

$$\sum_{k=p}^{q} (x_{k+1} - x_k) \leq (d-c) + 2\,\|\Delta\|\,.$$

Put $M = \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)| + 1$. Let $\varepsilon > 0$. By our hypotheses there is a cover $([c_j, d_j])_{j=1}^m$ of X of total length $< \frac{\varepsilon}{8M}$. Let Δ be a division of $[a,b]$ such that $||\Delta|| < \frac{\varepsilon}{16mM}$. When we compute

$$|UDS_\Delta(f) - UDS_\Delta(g)|$$

for all subintervals that are not intersecting X, f and g coincide and so the corresponding terms in the two sums cancel. Therefore

$$
\begin{aligned}
&|UDS_{\Delta}(f) - UDS_{\Delta}(g)| \\
&= \left| \sum_k (\sup_{x \in [x_k, x_{k+1}]} f(x) - \sup_{x \in [x_k, x_{k+1}]} g(x))(x_{k+1} - x_k) \right| \\
&\leq \sum_k \left| \sup_{x \in [x_k, x_{k+1}]} f(x) - \sup_{x \in [x_k, x_{k+1}]} g(x) \right| (x_{k+1} - x_k) \\
&\leq \sum_k (| \sup_{x \in [x_k, x_{k+1}]} f(x)| + | \sup_{x \in [x_k, x_{k+1}]} g(x)|)(x_{k+1} - x_k) \\
&\leq \sum_k (\sup_{x \in [x_k, x_{k+1}]} |f(x)| + \sup_{x \in [x_k, x_{k+1}]} |g(x)|)(x_{k+1} - x_k) \\
&\leq \sum_k (\sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |g(x)|)(x_{k+1} - x_k) \\
&= (\sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |g(x)|) \sum_k (x_{k+1} - x_k)
\end{aligned}
$$

where the sum is computed only over all subintervals intersecting X. This is definitely $\leq$ than the same sum computed over all subintervals intersecting the cover of

X and in this case, by a remark above,

$$\sum_k (x_{k+1} - x_k) < \sum_j (d_j - c_j) + \frac{\varepsilon}{8M} = \frac{\varepsilon}{4M}.$$

Thus

$$-\frac{\varepsilon}{4} < UDS_\Delta(f) - UDS_\Delta(g) < \frac{\varepsilon}{4}$$

A similar computation gives that

$$-\frac{\varepsilon}{4} < LDS_\Delta(f) - LDS_\Delta(g) < \frac{\varepsilon}{4}.$$

Let $\delta > 0$ such that $||\Delta|| < \delta$ implies that $UDS_\Delta(f) - LDS_\Delta(f) < \frac{\varepsilon}{2}$. Then, for $||\Delta|| < \min\{\delta, \frac{\varepsilon}{16mM}\}$,

$$\begin{aligned} UDS_\Delta(g) - LDS_\Delta(g) &< (UDS_\Delta(f) + \frac{\varepsilon}{4}) \\ &\quad + (-LDS_\Delta(f) + \frac{\varepsilon}{4}) \\ &= UDS_\Delta(f) - LDS_\Delta(f) + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, by the Darboux Criterion, g is Riemann integrable. Let $I = \int_a^b f(x)dx$. We have that

$$|UDS_\Delta(f) - I| \leq UDS_\Delta(f) - LDS_\Delta(f) < \frac{\varepsilon}{4}.$$

Then

$$\begin{aligned}|UDS_{\Delta}(g) - I| &\le |UDS_{\Delta}(g) - UDS_{\Delta}(f)| \\ &\quad + |UDS_{\Delta}(f) - I| \\ &\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

and hence $\int_a^b g(x)dx = I$. ■

Exercises

1. Prove that every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

2. Prove that every monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

3. Prove that the function

$$s_a(x) = \begin{cases} a & \text{if } x = 0 \\ \sin(1/x) & \text{if } x \in (0, 1] \end{cases}$$

is Riemann integrable.

4. Prove the theorem concerning the inheritance of integrability.

5. Prove the additivity property of Riemann integral.

References

[1] Constantin P. Niculescu, *An Introduction to Mathematical Analysis*, Universitaria Press, Craiova, 2005.

[2] W. Rudin: *Principles of Mathematical Analysis*, 3rd Edition, McGraw-Hill Book Co., New York, 1976.