Weyl's Ergodic Theorem

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Ergodic theory deals with the statistic behavior of averages and is (in many practical aspects), an extension of the measure theory. As an example of the power of this theory we will present a classical result of H. Weyl which has many applications in number theory.

Theorem (Weyl's Ergodic Theorem). *Let* α *be a real number such that* $\alpha/\pi \notin \mathbb{Q}$. *Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x+k\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,dt$$

for every continuous and periodic function $f : \mathbb{R} \to \mathbb{C}$, *of period* 2π, *and every* $x \in \mathbb{R}$.

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Proof. Let $x \in \mathbb{R}$. An easy computation shows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} e^{i\,n(x+k\alpha)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\,nt}\,dt$$

for all $n \in \mathbb{Z}$, which yields

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} P(x+k\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t)\,dt$$

for every trigonometric polynomial. We will extend this fact to each continuous and periodic function $f : \mathbb{R} \to \mathbb{C}$, of period 2π.

Let $\varepsilon > 0$. By the Weierstrass Approximation Theorem, there is a trigonometric polynomial $P_\varepsilon(x) = \sum_{n=-m}^{m} c_n e^{inx}$ such that $\sup\{|f(x) - P_\varepsilon(x)| : x \in \mathbb{R}\} < \varepsilon$. Therefore, the modulus of

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,dt - \frac{1}{N} \sum_{k=1}^{N} f(x+k\alpha)$$

is bounded above by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t) - P_\varepsilon(x)|\,dt \\ + \frac{1}{N}\sum_{k=1}^{N} |f(x+k\alpha) - P_\varepsilon(x+k\alpha)| \\ + \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_\varepsilon(t)\,dt - \frac{1}{N}\sum_{k=1}^{N} P_\varepsilon(x+k\alpha)\right|$$

from where the result follows. ■

Weyl's Ergodic Theorem can be easily extended to the case of continuous and periodic functions $f : \mathbb{R} \to \mathbb{C}$, of period $2T$, provided that $\alpha/T \notin \mathbb{Q}$.

Every continuous and periodic function $f : \mathbb{R} \to \mathbb{C}$, of period 1, is the extension by periodicity of a continuous function $g : [0,1] \to \mathbb{C}$ such that $g(0) = g(1)$. Therefore, Weyl's Ergodic Theorem can be reformulated as follows: *Suppose that* $x \in \mathbb{R}$ *and* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. *Then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\{x+k\alpha\}) = \int_0^1 f(t)\,dt \qquad (1)$$

for every continuous function $f : [0,1] \to \mathbb{C}$ *for which* $f(0) = f(1)$. Here $\{\cdot\}$ represents the fractional part.

It is worth to notice that the formula (1) can be extended to *all* Riemann integrable functions $f : [0,1] \to \mathbb{C}$.

The key remark: for any subinterval A of $[0,1]$ and for any $\varepsilon > 0$, there are two continuous functions $f, g : [0,1] \to \mathbb{C}$ with the following properties:

(a) $f(0) = f(1)$ and $g(0) = g(1)$.

(b) $f \leq \chi_A \leq g$.

(c) $\int_0^1 (g - f)\, dx < \varepsilon$.

Then

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N} f(\{x+k\alpha\}) &\leq \frac{1}{N}\sum_{k=1}^{N} \chi_A(\{x+k\alpha\}) \\ &\leq \frac{1}{N}\sum_{k=1}^{N} g(\{x+k\alpha\})\end{aligned}$$

and taking into account the inequalities

$$\int_0^1 f(t)\,dt \leq \int_0^1 \chi_A(t)\,dt \leq \int_0^1 g(t)\,dt \leq \int_0^1 f(t)\,dt+\varepsilon$$

we infer that (1) holds for the characteristic functions of subintervals of $[0,1]$. This conclusion extends by linearity to all step functions and thus to *all* Riemann integrable functions (due to Darboux Criterion of integrability).

As an application of this result we will determine the frequency of 7 showing up as the first digit of a power of 2 :

1, 2, 4, 8, 1, 3, 6, 1, ...

2^n starts with a 7 if there is some k such that

$$7\cdot 10^k \leq 2^n < 8\cdot 10^{k+1}.$$

Therefore $\log_{10} 7 + k \le n \log_{10} 2 < \log_{10} 8 + k + 1$ and thus $\{n \log_{10} 2\} \in [\log_{10} 7, \log_{10} 8)$. Since $\log_{10} 2 \notin \mathbb{Q}$, the frequency of 7 is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{[\log_{10} 7, \log_{10} 8)}(\{x + k \log_{10} 2\}) = \int_0^1 \chi_{[\log_{10} 7, \log_{10} 8)}(t)dt = \log_{10} 8 - \log_{10} 7 = \log_{10} \frac{8}{7}.$$

This may appear very surprising since the first encounter with 7 is for $2^{46} = 70\,368\,744\,177\,664$.

Exercises

1. Let A be a closed subset of $[0,1]$ which is not the whole interval. Find a function $f : [0,1] \to [0,1]$ such that $f(0) = f(1) = 0$, $f|_A = 0$ and f is not identical 0.

2. Let $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Prove, using the previous exercise and Weyl's Ergodic Theorem that the sequence $(\{n\alpha\})_n$ is dense in $[0,1]$.

3. Prove the continuous form of Weyl's Ergodic Theorem: *Let $\alpha \in \mathbb{R}$ be such that $\alpha/\pi \notin \mathbb{Q}$. Then*
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x+\alpha t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt$$
for every continuous and periodic function $f : \mathbb{R} \to \mathbb{C}$, of period 2π, and every $x \in \mathbb{R}$.

4. (H. Weyl). A sequence $(x_n)_n$ of elements of $[0,1]$ is called *uniformly distributed* if

$$\lim_{n\to\infty} \frac{1}{n} |\{k : 0 \leq k \leq n,\ x_k \in [a,b]\}| = b - a$$

for every subinterval $[a,b]$ (in other words, if the sequence visits every compact interval with a frequency equal to the size of the subinterval). Prove that the sequence $(x_n)_n$ is uniformly distributed if and only if it verifies one of the following equivalent conditions:

(a) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i m x_k} = 0$ for every $m \in \mathbb{Z}$;

(b) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x_k) = \int_0^1 f(x)dx$ for every $f \in C([0,1],\mathbb{R})$.

(c) If $A \subset [0,1]$ and its characteristic function is Riemann integrable then

$$\lim_{n\to\infty} \frac{1}{n} |\{k : 0 \leq k \leq n,\ x_k \in A\}| = m(A)$$

where $m(A)$ denotes the Lebesgue measure of A.

5. Prove that every uniformly distributed sequence in [0,1] is dense. Retrieve the result that makes the objective of Exercise 2.

6. Give an example of sequence dense in [0,1] which is not uniformly distributed.

7. Suppose that an oscillatory system evolves according to the law $f(t) = A \sin \omega t$ (where $\omega > 0$, $\omega/2\pi \notin \mathbb{Q}$, $A > 0$). We measure the values of f at the moments $t = 0, 1, 2, \ldots$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k : 0 \le k \le n, \;\; |f(k)| < 10^{-2} \right\} \right| \\ = \frac{2}{\pi} \arcsin 10^{-2} A^{-1}.$$

Hint: $|f(k)| < 10^{-2}$ is equivalent to $|\omega n - k\pi| < \arcsin 10^{-2} A^{-1}$ i.e., $\frac{\omega n}{\pi}$ belongs to

$$(0, \frac{1}{\pi} \arcsin 10^{-2} A^{-1}) \cup \left(1 - \frac{1}{\pi} \arcsin 10^{-2} A^{-1}, 1 \right)$$

$\mathrm{mod}\, 1$. As $\frac{\omega}{\pi}$ is irrational, the sequence $(\frac{\omega n}{\pi})_n$ is uniformly distributed.

References

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[3] H. Weyl, Über die Gleichverteilung von Zählen mod. Eins, *Math. Ann.* **77** (1916), pp. 313-352.