Construction of elementary functions

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My lecture was an overview of Chapter 6 in [1].

Exercises

1. Recall the definition of the exponential of base a : *If a is a real number, $a > 1$, then there is a unique function $f : \mathbb{R} \to \mathbb{R}$ such that:*

   (a) $f(x + y) = f(x)\, f(y)$ *for all* $x, y \in \mathbb{R}$;

   (b) $f(1) = a$;

   (c) *The function f is increasing*.

   See [1]. Prove that e^x establishes a homeomorphism between $\mathbb{R}$ and the interval $(0, \infty)$.

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2. Prove the formula

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function, continuous at a point $x = x_0$. Prove that f is continuous everywhere (and thus it is of the form $f(x) = ax$).

4. Prove that the power functions x^a (for $a \in \mathbb{R}$) are the only solutions $f : (0, \infty) \to \mathbb{R}$ of the functional equation

$$f(xy) = f(x)f(y) \quad \text{for all } x, y > 0,$$

which are continuous and non identically zero.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + y) = f(x) f(y)$ for all $x, y \in \mathbb{R}$. Prove that either f is identically zero, or there is $a \in \mathbb{R}$ such that $f(x) = e^{ax}$.

6. Infer from the preceding exercise the power series expansion of the exponential:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \quad \text{for } x \in \mathbb{R}.$$

7. Suppose that the sine function is introduced (in the complex domain) via the formula

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots .$$

Prove that the restriction of the sine function to the real axis is bounded and

$$|\sin x - \sin y| \leq |x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

References

[1] Constantin P. Niculescu, *An Introduction to Mathematical Analysis*, Universitaria Press, Craiova, 2005.