Math Track
School of Mathematical Sciences
Volume 1, Number 1, Pages 31 – 40
ISSN: 1818-5495 (Online)

PRIME NUMBERS AND RIEMANN ZETA FUNCTION

AMER IQBAL

Department of Mathematics,
University of Washington,
Seattle, WA 98195, U.S.A.
E-mail: iqbal@math.washington.edu

Abstract

A brief introduction to the relation between prime counting function and the zeroes of the Riemann zeta function is provided.

1. Introduction

The distribution of prime numbers, numbers with only two positive divisors, among the natural numbers has many interesting properties. The distribution seems follows no regular pattern but it can be seen that the prime numbers become less common as we go towards large numbers. However, it is also known that they sometimes occur in pairs called the twin primes. Bernhard Riemann, a 33 years old German mathematician, studied the distribution of prime numbers and proved that the distribution of prime numbers is very closely related to the complex zeroes of a function now known as the Riemann zeta function,

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, s \in C \setminus \{1\} \tag{1}$$

Riemann's famous hypothesis asserts that all complex zeroes of $\zeta(s)$ lie on a straight line. More precisely the complex zeroes of the zeta function have real part 1/2. This has been checked for the first 10 trillion zeros [1], but no general proof is known. A proof of the hypothesis will give greater understanding of the distribution of the prime numbers. ([2] is an excellent popular book on Riemann hypothesis. A more technical account is given in [3]). Prime numbers have been a source of constant fascination for the mathematicians. The first proof of the fact that there is infinite number of primes was given by Euclid in the third century BC. A major breakthrough in the understanding of the distribution of prime numbers came in 1896, when Jacques Hadamard and Charles de la Valle Poussin independently proved the Prime Number theorem [4, 5]. Let us define the prime counting function as follows,

$$\pi(x) = \sum_{p \le x} 1 = \text{ \# of primes less than or equal to } x. \qquad (2)$$

The graph of $\pi(x)$ for various intervals is shown in Fig. 1, Fig. 2 and Fig. 3. Since there are infinitely many prime numbers it is clear that

$$\pi(x) \mapsto \infty \; as \; x \mapsto \infty \qquad (3)$$

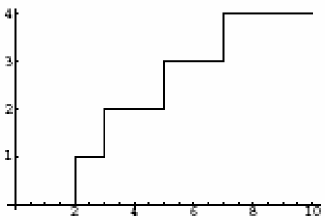


Figure 1: $\pi(x)$ for $x \le 10$

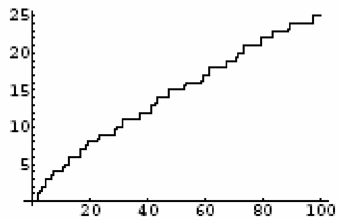


Figure 2: $\pi(x)$ for $x \le 100$.

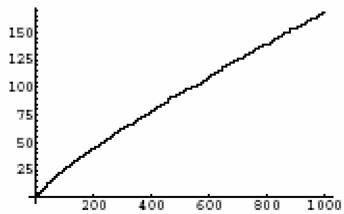


Figure 3: $\pi(x)$ for $x \le 1000$.

But how exactly $\pi(x)$ approaches infinity for x very large was not known for a long time. The prime number theorem, proposed by Gauss in 1792, gives an asymptotic formula for $x \mapsto \infty$,

$$\pi(x) \approx \frac{x}{\ln(x)} \qquad (4)$$

Gauss later refined this to

$$\pi(x) \approx Li(x) := \int_2^x \frac{dt}{\ln(t)} . \tag{5}$$

2. Riemann zeta function:

The Riemann zeta function was actually first studied by Euler for real s. He also gave a product representation of the zeta function, which shows a clear link with the prime numbers. Recall that for natural numbers the unique factorization property holds i.e., if n is a natural number then it has a unique representation in terms of primes given by

$$n = \prod_{i=1}^{\infty} p_i^{a_i} . \tag{6}$$

Where the product is over all prime numbers and $a_i s$ are non-negative integers such that most of them are actually zero. Using this property of natural numbers the zeta function can be expressed as a product, known as the Euler product,

$$\varsigma(s) = \prod_{i=1}^{\infty} (1 - p_i^{-s})^{-1} \tag{7}$$

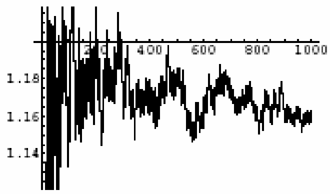


Figure 4: $\frac{\pi(x) ln(x)}{x}$ for $2 \leq x \leq 1000$.

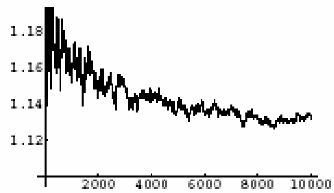


Figure 5: $\frac{\pi(x) ln(x)}{x}$ for $2 \leq x \leq 10000$.

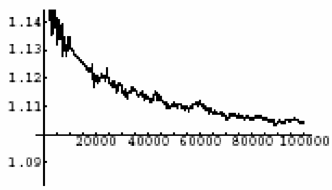


Figure 6: $\frac{\pi(x) ln(x)}{x}$ for $2 \leq x \leq 100000$.

Riemann rediscovered the zeta function while studying the distribution of primes. He studied it as a function of a complex variable and defined an analytic continuation to the whole complex plane. From the Euler product representation it is clear that there are no zeroes of the zeta function for $Re(s) > 1$.

3. Analytic Continuation and Functional Equation:

The Riemann zeta function as defined in the last section converges only for $Re(s) > 1$. However, it can be analytically continued so that it converges for $Re(s) > 0$ (except for $s = 1$) using the following formula:

$$\varsigma(s) = \frac{\eta(s)}{2^{1-s}-1} \qquad , \qquad \eta(s) = \sum_{n=1}^{\infty}(-1)^n n^{-s} \tag{8}$$

The function $\eta(s)$ is convergent for $Re(s) > 0$. Actually it turns out that the Riemann zeta function has an analytic continuation to the whole complex plane given by

$$\varsigma(s) = \frac{\Gamma(1-s)}{2\pi i}\oint_C \frac{u^{s-1}}{e^{-u}-1}du\,, \tag{9}$$

where the contour C is a path coming from $-\infty$ just below and parallel to the real axis, circling the origin anticlockwise and returning to $-\infty$ parallel and just above the real axis. The Gamma function, $\Gamma(s)$ is defined as

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt \tag{10}$$

The above integral gives a definition of the zeta function valid for all $s \neq 1$. $\varsigma(s)$ has a pole at $s = 1$ with residue 1. It can be expanded in a Laurent series around $s = 1$,

$$\varsigma(s) = \frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + \gamma_2(s-1)^2 + \ldots\ldots.. \tag{11}$$

Where

$$\gamma_k = \frac{(-1)^k}{k!} Lim_{N\to\infty}(\sum_{m\le N} \frac{\ln^k(m)}{m} - \frac{\log^{k+1} N}{k+1}) \qquad (12)$$

γ_0 is the famous Euler's constant equal to 0.5772157...

The Riemann zeta function satisfies a remarkable identity, discovered by Riemann, known as the functional equation of the Riemann zeta function. It is given by

$$\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\varsigma(1-s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\varsigma(s) \qquad (13)$$

The functional equation given above can be proved using the modular property of the theta function $\theta(t) = \sum_{n=-\infty}^{+\infty} e^{i\pi n^2 T}$,

$$\theta(T) = \frac{1}{\sqrt{-iT}}\theta(-\frac{1}{T}) \qquad (14)$$

and the following representation of the zeta function valid for $Re(s) > 1$,

$$\varsigma(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}\int_0^\infty (\frac{\theta(it)-1}{2}) t^{\frac{s}{2}-1} dt \qquad (15)$$

4. Trivial and Non-trivial zeroes:

From the functional equation of the Gamma function it follows that $\varsigma(s)$ is zero at $s = -2n$, n =1, 2, 3… To see this let $s = -2n$ in the functional equation, eq (13) to obtain

$$\varsigma(-2n) = \pi^{-2n-\frac{1}{2}} \frac{\Gamma(n+\frac{1}{2})\varsigma(2n+1)}{\Gamma(-n)} \qquad (16)$$

since $\Gamma(n+\frac{1}{2})$ and $\varsigma(2n+1)$ are finite and $\Gamma(s)$ has a pole at $s = -n$ therefore $\varsigma(-2n) = 0$ for n=1, 2, 3 ... These zeroes on the real axis are called the trivial zeroes.

The complex zeroes of the zeta function are more interesting and, as we will see in the next section, are directly related to $\pi(x)$, the prime counting function. From the functional equation it is clear that if $s = s_0$ is a complex zero of $\varsigma(s)$ then $1 - s_0$ is also a zero. Since $\overline{\varsigma(s)} = \varsigma(\overline{s})$ therefore $s = \overline{s_0}$ is a complex zero if s_0 is a complex zero. Riemann conjectured that the complex zeroes of the zeta function lie on a straight line, critical line, given by $\mathrm{Re}(s) = 1/2$. Thus according to Riemann hypothesis if $\varsigma(s) = 0$ and $\mathrm{Im}(s) \neq 0$ then $s = \frac{1}{2} + iw, w \in R$.

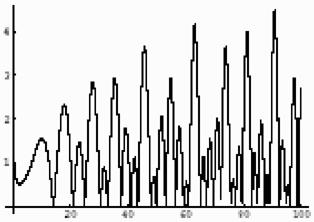


Figure 7: $|\zeta(\frac{1}{2} + i\gamma)|$ for $0 \leq \gamma \leq 100$.

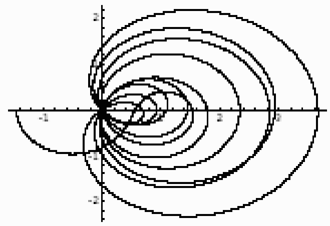


Figure 8: $\zeta(\frac{1}{2} + i\gamma)$ for $0 \leq \gamma \leq 50$.

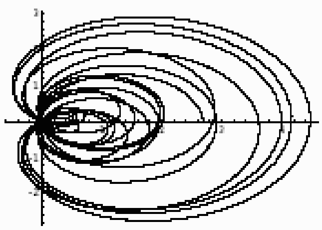


Figure 9: $\zeta(\frac{1}{2} + i\gamma)$ for $50 \leq \gamma \leq 100$.

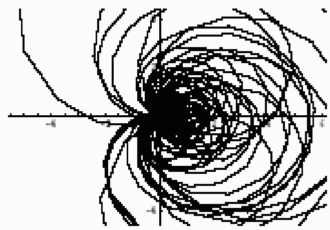
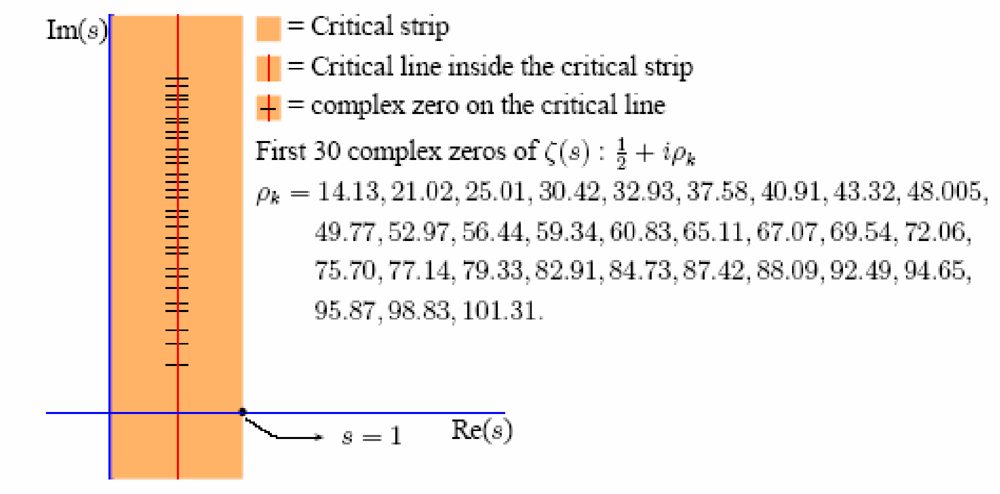


Figure 10: $\zeta(\frac{1}{2} + i\gamma)$ for $100000 \leq \gamma \leq 100050$.

The table below gives the first ten non-trivial zeroes of the zeta function.



It was shown recently [6] that Riemann hypothesis is equivalent to proving the following relation between the harmonic numbers and the divisor sums

$$\sigma(n) \leq H_n + in(H_n)e^{H_n}, n \in \{1, 2, 3, ...\} \tag{17}$$

where $\sigma(n) = \sum_{d \backslash n} d$ *and* $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the harmonic number.

Euler's formula gives a product representation of the zeta function. Another product representation involving the zeros of the zeta function is the Hadamard product given by

$$\varsigma(s) = \frac{e^{(\ln(2\pi)-1-\frac{\gamma_0}{2})s}}{2(s-1)\Gamma(1+s/2)} \prod_{\substack{\varsigma(w)=0 \\ \mathrm{Im}(w)>0}} (1-\frac{s}{w})e^{\frac{s}{w}} \tag{18}$$

where the product is over w, the non-trivial roots of the zeta function. The non-trivial roots of the zeta function; w_k, can be used to define series similar to the Harmonic series which in this case are actually convergent. Define

$$Z(n) := \sum_k w_k^{-n},$$

then $Z(n)$ is convergent and $Z(1)$ was determined by Riemann to be $\frac{1}{2}(2+\gamma_0 - \ln(4\pi))$.

5. $\pi(x)$ and complex zeros of $\varsigma(s)$:

To understand the relation between the prime counting function and the zeros of the zeta function let us define an arithmetic function which counts, with a certain weight, primes and powers of primes less than *x*,

$$J(x) = \sum_{r=1}^{\infty} \sum_{p^r \le x} \frac{1}{r} \tag{19}$$

The function $J(x)$ can be expressed in terms of the prime counting function,

$$J(x) = \sum_{r=1}^{\infty} \sum_{p^r \le x} \frac{1}{r} = \sum_{r=1}^{\infty} \sum_{p \le x^{1/r}} \frac{1}{r} = \sum_{r=1}^{\infty} \frac{1}{r} \sum_{p \le x^{1/r}} 1 = \sum_{r=1}^{\infty} \frac{1}{r} \pi(x^{1/r}) \tag{20}$$

Using the Mobius inversion formula it is possible to express $\pi(x)$ in terms of $J(x)$,

$$\pi(x) = \sum_{r=1}^{\infty} \frac{\mu(r)}{r} J(x^{1/r}) \tag{21}$$

The link between $\pi(x)$ and the zeroes of the zeta function are provided by the function $J(x)$ and the above expression of $\pi(x)$ in terms of $J(x)$. To see this let us take the natural log of the Euler product representation of $\varsigma(s)$,

$$\varsigma(s) = \prod_{p=prime} (1-p^{-s})^{-1} \tag{22}$$

$$\ln \varsigma(s) = -\sum_{p=prime} \ln(1-p^{-s}) = \sum_{p=primes} \sum_{k=1}^{\infty} \frac{p^{-ks}}{k} \tag{23}$$

$$= s \sum_{p=primes} \sum_{k=1}^{\infty} \frac{1}{k} \int_{p^k}^{\infty} x^{-s-1} dx = s \sum_{k=1}^{\infty} \frac{1}{k} \sum_{prime} \int_{p^k}^{\infty} x^{-s-1} dx$$

$$= s \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{\infty} \int_{p_i^k}^{\infty} x^{-s-1} dx$$

Since the above integral jumps every time a prime power occurs therefore,

$$\frac{\ln \varsigma(s)}{s} = \int_{0}^{\infty} J(x) x^{-s-1} dx \qquad (24)$$

Thus we see that $\frac{\ln \varsigma(s)}{s}$ is the Mellin transform of $J(x)$. The inverse Mellin transform gives $J(x)$ in terms of $\frac{\ln \varsigma(s)}{s}$,

$$J(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\ln \varsigma(s)}{s} x^{-s} ds\,. \qquad (25)$$

Using the product representation eq (18), of the zeta function in terms of its roots we get,

$$J(x) = li(x) - \sum_{w} li(x^{w}) - \ln(2) + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\ln(t)}\,. \qquad (26)$$

Where $li(x) = \int_{\mu}^{x} \frac{du}{\ln(u)}$ and $\mu = 1.45136...$ is the Soldner's constant. Since $\pi(x)$ is determined by $J(x)$ therefore we see that the prime counting function depends on the zeroes of the zeta function. This was the famous result proposed by Riemann and proved by Mangoldt in 1895.

References:

[1] X. Gourdon, "***The 1013 first zeros of the Riemann zeta function, and zeros computation at very large height***", http://numbers.computation.free.fr /Constants/ Miscellaneous/zetazeros1e13.1e24.pdf.

[2] J. Derbyshire, "***Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics***", Penguin Books Ltd 2003.

[3] J. B. Conrey, "***The Riemann Hypothesis***", Notices of the AMS, March 2003.
[4] E. C. Titchmarsh, ***"The Theory of the Riemann Zeta-function"***, Oxford University Press 2003.
[5] A. Ivic, ***"The Riemann Zeta-Function: Theory and Applications"***, Dover Publications Inc 2003.
[6] Jeffrey C. Lagarias, "***An Elementary Problem Equivalent to the Riemann Hypothesis***", http://arxiv.org/abs/math. NT/0008177/.