

MATHEMATICAL MODELING WITH DIFFERENTIAL EQUATIONS

SAIFULLAH

*School of Mathematical Sciences, Government College University,
68-B New Muslim Town, Lahore, Pakistan.
E-mail: saifullahkhalid75@yahoo.com*

Abstract

This article aims to provide the students some basic modeling skills, which will have applications to a wide variety of problems. The process of constructing a model is carefully considered and presented in full details. Flow diagrams and word equations¹ are constructed both to aid in the model building process and to develop the mathematical equations. Graphical methods are used and it is the intention of this article to provide an interactive use of software.

The main mathematical technique used is that of solving differential equations. An example of differential equation model in which two quantities interact with each other is shown. This model is simple enough and is governed by two simultaneous first order differential equations.

1. General principles of mathematical modeling

Scientists, engineers and economists, working on a wide variety of problems, now a day find it useful to setup mathematical models of the systems, which they are investigating. To do this, they give a simplified description of the problem, which allows equations to be setup and then solved to make predictions. By studying this article one will appreciate the extent to which mathematical modeling is currently being used in the various sciences and why mathematics is so useful in helping to solve scientific problems.

1.1. Use of models

Models of systems have become part of our everyday lives. They range from global decisions having a profound impact on our future, to local decisions about whether to cycle to university based on weather predictions. Together with their

¹ Word equation: which relates quantities expressed in words before assigning mathematical symbols to them.

provision of a deeper understanding of the processes involved, this predictive nature of models, which aids in decision-making, is of their key strengths.

One of the aims of modeling is to segregate the most important factors in a problem and ignore those, which may not be important. Even very complicated processes can initially be analyzed using very simple mathematical models, which may later be extended to more complex and realistic models by incorporating more features.

1.2. Diverse applications

It is impossible to imagine modern science without the wide application of mathematical modeling. Mathematical modeling is a bridge between the study of mathematics and the applications of mathematics to various fields.

In particular, many processes can be described with mathematical equations; that is, by mathematical models. Such models have a use in a diverse range of disciplines.

In general, with such a diversity of modeling applications it is not surprising that many different areas of mathematics are used. These include differential and integral calculus, differential equations, matrices, statistics, probability, to name just a few.[4]:

1.3. Decision-making

To build a model there are many decisions that must be made, either explicitly or more often implicitly. Some of these are shown in the following Fig.1.[3]:

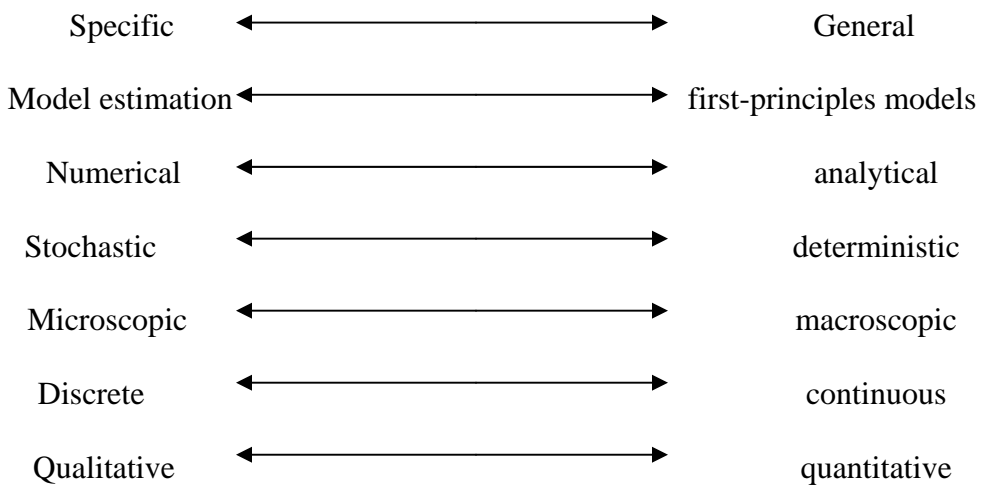


Fig 1: Some levels of description for mathematical model building

Each of these is a continuum rather than a discrete choice. This list is not exhaustive, but it's important to keep returning to it; many efforts fail because of an unintentional attempt to describe either too much or too little.

Modeling is an extremely powerful tool, a framework for research, debate and planning, which provides a valuable source of information for decision-making.

1.4. Problem solving process

Following are the steps of the problem solving process.

- 1) Problem identification.
- 2) Model construction or selection.
- 3) Identification and collection of data.
- 4) Model validation.
- 5) Calculation of solutions to the model.
- 6) Model implementation and maintenance.

1.5. Constructing model

A mathematical model is a simplification of a complex real world problem, which is cast into the form of mathematical equations. There are several steps to the construction of a mathematical model, which are as follows:[5]:

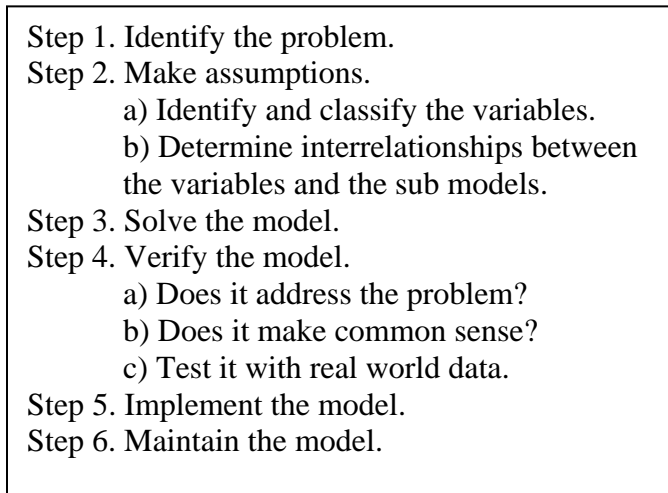


Fig 2: Construction of a mathematical modeling

1.6. The iterative nature of model construction

It is clear from the following figure that model construction is of iterative nature [5]:

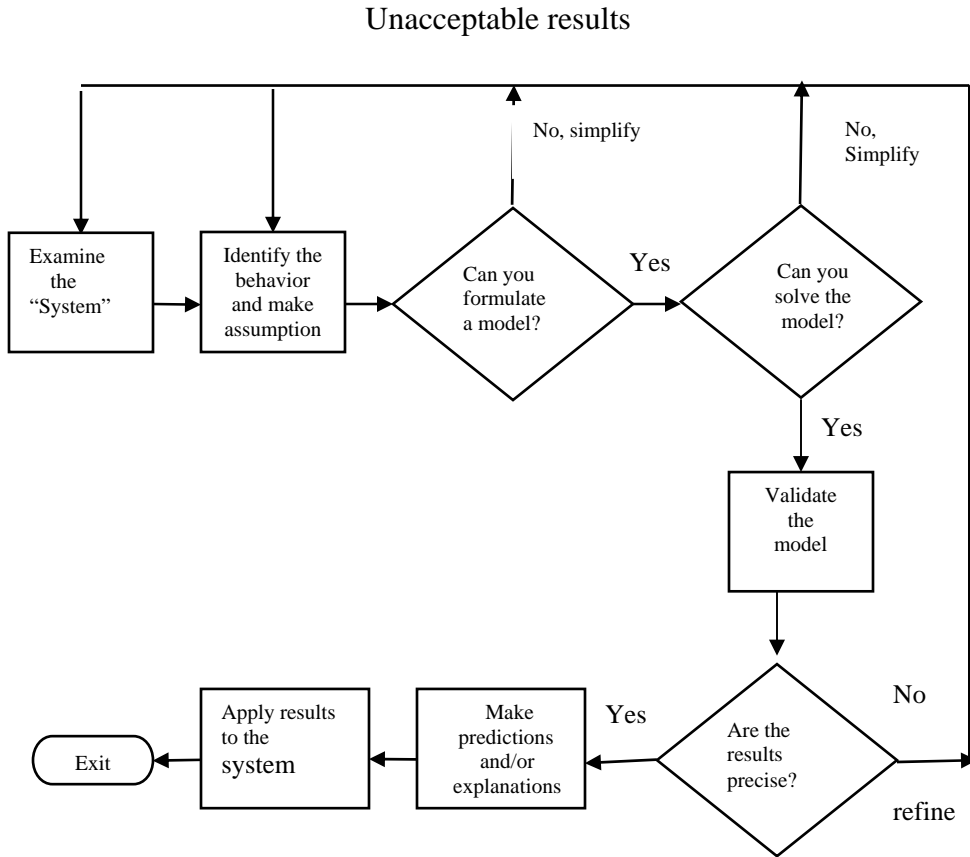


Fig 3: The iterative nature of model construction

2. Example: The model of the battle between two opposing groups

Battles between armies have fought since antiquity. In ancient times battles were primarily hand-to-hand combat. With the development of archery and then gunpowder a crucial feature of battles has been aimed fire. Although many factors can affect the outcome of a battle, experience has shown that numerical superiority and superior military training are critical.

The combat situation of two groups fighting a battle can be modeled, subject to some simplifying assumptions, as a pair of coupled differential equations. Mathematical models of combat can be used to understand what factors can

influence the outcome of the battle: some questions which might be asked include which side is the victor, how many survivors remain, how long does the battle take?

In this example we look at one particular combat situation where one group is exposed to fire and the other is hidden. This situation may be used to model guerilla warfare. The exposed group will be termed the 'regular army' and the hidden group the 'guerilla army'. We will use the following notations

$$x = \{ \text{number of regular army's soldiers at time } t \}$$

$$y = \{ \text{number of guerilla army's soldiers at time } t \}$$

and assume that the number of soldiers can be approximated as continuous variables.

In an isolated battle the major factor reducing the size of each army is the number of soldiers put out of action (killed, wounded or taken prisoner) by the opposing army. First we make some basic assumptions and then develop the model based on them.

- i) we assume the number of soldiers to be sufficiently large so that we can neglect random differences between them.
- ii) We also assume that there are no reinforcements and no operational losses (i.e. due to desertion or disease).
- iii) Neither army takes prisoners.
- iv) Each army is using gunfire against the other.

These are assumptions that can easily be relaxed at a later stage if the model is inadequate.

Thus with δx and δy denoting the changes in the numbers of the respective armies during a time interval δt ,

$$\delta x = - \{ \text{number of regular army's soldiers hit by gunfire of guerilla army} \}$$

$$\delta y = - \{ \text{number of guerilla army's soldiers hit by gunfire of regular army} \}.$$

The number of soldiers hit in small time interval δt , is equal to the product of

- (i) the rate at which each soldier shoots (R_x shots per unit time for the regular army and R_y for the guerilla army),
- (ii) the probability that a single shot hits its target (P_x for the regular army and P_y for the guerilla army),
- (iii) the number of soldiers firing the shots.

Hence

$$\delta x = -R_y P_y y \delta t \quad \text{and} \quad \delta y = R_x P_x x \delta t \quad (1)$$

The firing rates R_x and R_y are assumed to be constant, while the probabilities P_x and P_y are determined according to whether the target is exposed or hidden.

If the target is exposed (the regular army soldiers), it is reasonable to assume that each single shot has a constant probability of hitting its target, independent of the number of regular soldiers. So P_y is a constant with respect to x and y .

If the target is hidden, however, then the probability of hitting a soldier by a shot fired at random into a given area will depend on the concentration of hidden soldiers in the area. If there are y guerilla soldiers and if each guerilla soldier has on average an area α exposed, then $P_x = \frac{\alpha}{A} y$, where A is the total area occupied by the guerilla soldiers. So αy gives the total area of soldiers available to be hit by random fire.

Substituting this formula into (1) and dividing by δt gives the coupled non-linear equations.[2]

$$\begin{cases} \frac{dx}{dt} = -by \text{ ----(i)} \\ \frac{dy}{dt} = -axy \text{ ----(ii)} \end{cases} \quad (2)$$

$$\text{Where } b = R_y P_y, a = R_x \frac{\alpha}{A},$$

Here ‘ a ’ and ‘ b ’ are positive constants which we may characterize as the rates of losses due to rival actions.

Now solving equations in (2):

$$\begin{aligned} \text{(i)} \Rightarrow x \frac{dx}{dt} &= -bxy \\ &-\frac{x}{b} \frac{dx}{dt} = xy \\ \text{(ii)} \Rightarrow -\frac{1}{a} \frac{dy}{dt} &= xy \end{aligned}$$

Equating the above two equations, we get

$$\begin{aligned} -\frac{x}{b} \frac{dx}{dt} &= -\frac{1}{a} \frac{dy}{dt} \\ ax \frac{dx}{dt} - b \frac{dy}{dt} &= 0 \\ \frac{d}{dt} \left(a \frac{x^2}{2} - by \right) &= 0 \\ \frac{a}{2} x^2(t) - by(t) &= C \end{aligned} \quad (3)$$

OR
$$y(t) = \frac{1}{b} \left(\frac{a}{2} x^2(t) - C \right)$$

Where C is constant depending on the initial conditions.

$$\begin{aligned} \text{at } t = 0, \quad \frac{a}{2} x^2(0) - b y(0) &= C & \text{OR} & \quad \frac{a}{2} x_0^2 - b y_0 = C \\ \implies \frac{a}{2} x^2(t) - b y(t) &= \frac{a}{2} x^2(0) - b y(0) = C & & \quad (4) \end{aligned}$$

Now we can study the phase trajectories of the system (1) and (2) with the help of equation (4). The parabolas defined by (3) for varying C are sketched in Fig.4, only the region of the phase-plan $x, y \geq 0$ is sketched since the number of soldiers cannot be negative. Also from (2), if $x, y > 0$, then $\frac{dx}{dt}, \frac{dy}{dt} < 0$, so all the trajectories point towards the axes.

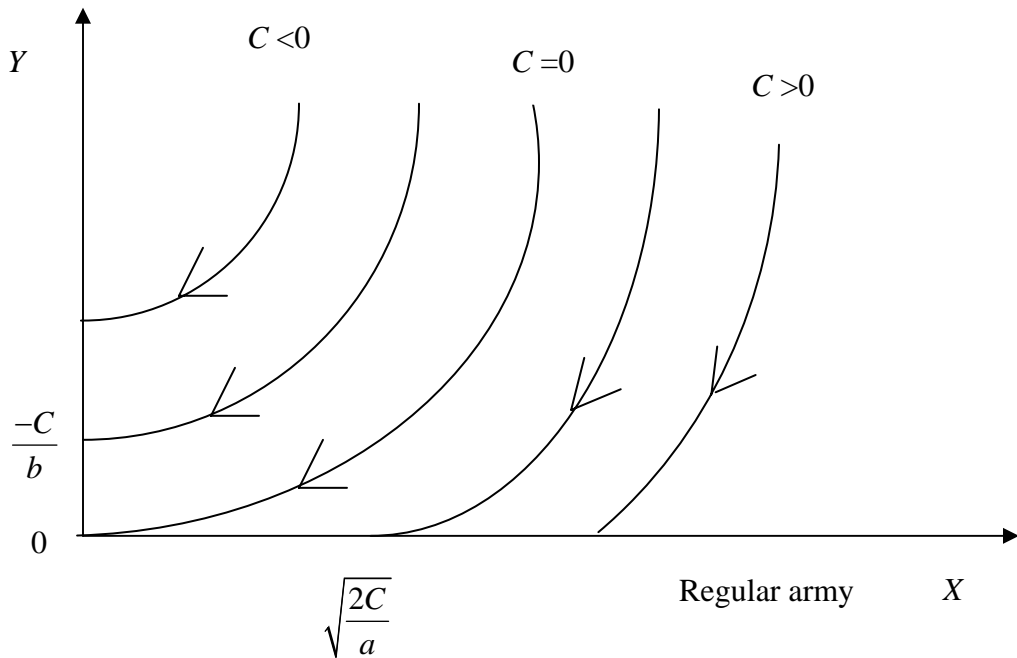


Fig 4: Phase Portrait

Now from equation (3), we eliminate $y(t)$ using (i), then we have

$$\frac{ax^2}{2} + \frac{dx}{dt} = C \Rightarrow \frac{dx}{dt} = C - \frac{ax^2}{2}$$

Then we solve this equation for different values of C .

Case 1: When $C = 0$,

$$\begin{aligned} \frac{dx}{dt} &= -\frac{ax^2}{2} \\ \frac{dx}{x^2} &= \frac{-a}{2} dt \\ \frac{-1}{x} &= \frac{-at}{2} - C_1 \\ x &= \frac{2}{at + 2C_1} \\ x(t) &= \frac{2}{at + 2C_1} \end{aligned}$$

It is clear from the above equation that

$$x \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Thus when $C = 0$, then $\frac{y(0)}{x^2(0)} = \frac{a}{2b}$, and mutual annihilation will occur

So there is no victory in this case and it is shown in Fig.5.

Now we present the plot $x(t)$ in Maple computer algebra system.[7]:

```
> x:=2/(a*t+2*c);
> a:=1: c:=1:
> plot(x,t=0..100);
```

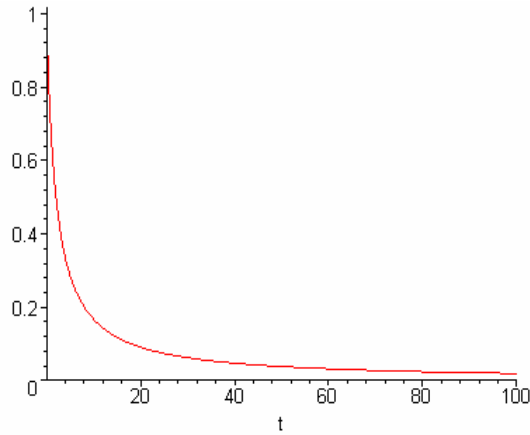


Fig 5: No victory in this case

Case 2: When $C > 0$.

$$\frac{dx}{dt} = C - \frac{ax^2}{2}$$

$$\frac{\frac{2}{a} dx}{\frac{2C}{a} - x^2} = dt$$

and, thus,

$$\frac{1}{\sqrt{2aC}} \ln \left| \frac{\sqrt{\frac{2C}{a}} + x}{\sqrt{\frac{2C}{a}} - x} \right| = t + C_2$$

$$\left| \frac{\sqrt{\frac{2c}{a}} + x}{\sqrt{\frac{2c}{a}} - x} \right| = e^{\sqrt{2aC}(t+C_2)}$$

Let $\alpha = \sqrt{\frac{2C}{a}}$, then $\left| \frac{\alpha + x}{\alpha - x} \right| = \begin{cases} \frac{\alpha + x}{\alpha - x}, & x \in [\alpha, \infty) \\ \frac{x + \alpha}{x - \alpha}, & x \in [0, \alpha) \end{cases}$

For $x \in [\alpha, \infty)$, we have

$$x(t) = \sqrt{\frac{2c}{a} \frac{e^{\sqrt{2aC}(t+C_2)} - 1}{e^{\sqrt{2aC}(t+C_2)} + 1}}$$

Now clearly from above equation we conclude that

$$x \rightarrow \sqrt{\frac{2c}{a}} \text{ as } t \rightarrow \infty.$$

i.e., the regular army wins. It is shown in Fig.6.

For $x \in [0, \alpha)$, we have

$$x(t) = \sqrt{\frac{2c}{a} \frac{e^{\sqrt{2aC}(t+C_2)} + 1}{e^{\sqrt{2aC}(t+C_2)} - 1}}$$

And clearly from above equation we conclude that

$$x \rightarrow \sqrt{\frac{2c}{a}} \text{ as } t \rightarrow \infty,$$

i.e., the regular army wins.

Also here $\frac{dx}{dt} \rightarrow 0$

So from (i) $\Rightarrow y \rightarrow 0$ as $t \rightarrow \infty$.

Hence in this case regular army wins and it is shown in Fig.6.

Now we present the graph $x(t)$ in Maple computer algebra system.[7]:

```
>x := (2*c/a)^0.5*(exp((2*a*c)^0.5*(t+c2))-1)/(exp((2*a*c)^0.5*(t+c2))+1);
```

$$x := 1.414213562 \frac{\left(\frac{c}{a}\right)^{.5} \left(e^{(1.414213562(ac)^{.5}(t+c2))} - 1\right)}{e^{(1.414213562(ac)^{.5}(t+c2))} + 1}$$

```
> c:=1: a:=1: c2:=1:
> plot(x,t=0..30);
```

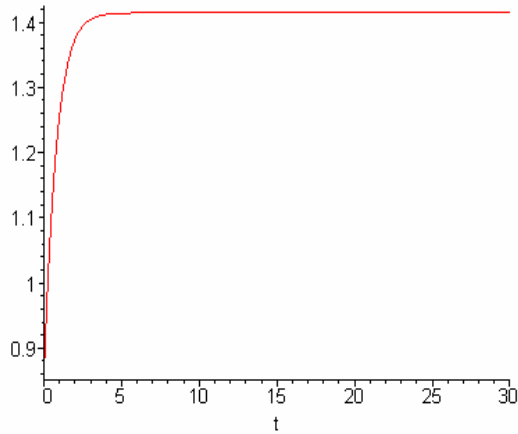


Fig 6: Regular army wins in this case

Case 3: When $C < 0$

$$\frac{dx}{dt} = C - \frac{ax^2}{2}$$

$$\frac{dx}{dt} = -d - \frac{ax^2}{2},$$

(where $C = -d$ and $d > 0$).

$$\frac{dx}{\frac{2d}{a} + x^2} = -\frac{a}{2} dt$$

$$\frac{1}{\sqrt{\frac{2d}{a}}} \tan^{-1} \left(\frac{x}{\sqrt{\frac{2d}{a}}} \right) = -\frac{a}{2} t + C_3$$

$$x(t) = \sqrt{\frac{2d}{a}} \tan \left(\sqrt{\frac{2d}{a}} \left(C_3 - \frac{a}{2} t \right) \right) \tag{5}$$

Graph of (5) is shown in Fig.7.

Now let $t = t_1$. We substitute $x(t_1)=0$ and find t_1 .

$$0 = \sqrt{\frac{2d}{a}} \tan\left(\sqrt{\frac{2d}{a}}\left(C_3 - \frac{a}{2}t_1\right)\right)$$

$$\tan\left(\sqrt{\frac{2d}{a}}\left(C_3 - \frac{a}{2}t_1\right)\right) = 0, \quad \text{as } d > 0 \text{ and } a > 0.$$

$$\sqrt{\frac{2d}{a}}\left(C_3 - \frac{a}{2}t_1\right) = n\pi, \quad n = 0, 1, 2, 3, \dots$$

$$t_1 = \frac{2C_3}{a} - \sqrt{\frac{2}{ad}}n\pi$$

$$t_1 = \frac{2C_3}{a} - \sqrt{-\frac{2}{aC}}n\pi, \quad \text{as } d = -C.$$

We know that $\frac{dx}{dt} = C - \frac{ax^2}{2}$

$$(i) \Rightarrow -by = C - \frac{ax^2}{2}$$

$$\text{when } x(t_1) = 0 \Rightarrow -by = C$$

$$\text{or } y = -\frac{C}{b}$$

$$\text{so } y(t) = -\frac{C}{b} > 0, \quad \text{as } C < 0.$$

Hence for $C < 0$ and at the moment $t = t_1$ regular army vanishes but guerilla army's people are there, so in this case guerilla army win.

Now we present the graph $x(t)$ in Maple computer algebra system.[7]:

>x := (2*d/a)^0.5*tan((2*d/a)^0.5*(c3-a*t/2));

$$x := 1.414213562\left(\frac{d}{a}\right)^{.5} \tan\left(1.414213562\left(\frac{d}{a}\right)^{.5} \left(c3 - \frac{1}{2}at\right)\right)$$

```
>d:=2: a:=1: c3:=1:
>plot(x,t=10..12);
```

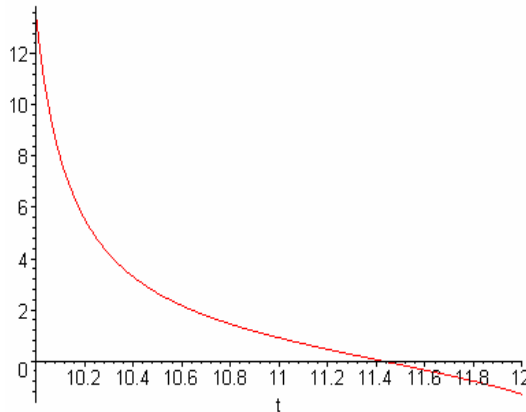


Fig 7: The guerilla army wins in this case.

Conclusion.

Thus for $C > 0$ regular army wins, for $C < 0$ guerilla army wins and for $C = 0$ a mutual annihilation will occur.

However, if the guerilla army were to double the number of soldiers it had initially we see that the regular army would only have to increase the size of its army by a factor of $\sqrt{2}$ to match the guerilla army.

Generally, however if the guerilla army often has an advantage over the regular army. If the guerilla army is spread over a very large area A then the parameter $a = R_x \frac{\alpha}{A}$, is small number compared with $b = R_y P_y$. Thus (4) states that the two armies are evenly matched when $x(0) = x_0$ is large and $y(0) = y_0$ is relatively small.

I am a great admirer of real mathematicians, who I define to be people who always are aware of the difference between what they know and what they think they know; I certainly am not one. I hope that my presumption in reducing whole discipline to ten or so pages apiece of essential ideas is exceeded by the value of such a compact presentation. For my inevitable sins of omission, commission, and every thing in between, I welcome your feedback at saifullahkhalid75@yahoo.com.

References.

- [1] A.Kondratyev. Lectures on Ordinary Differential equations at School of Mathematical Sciences: (Fall semester-2004).
- [2] A.Kondratyev. Lectures on Mathematical Modeling at School of Mathematical Sciences: (Spring semester-2005).
- [3] A.Kondratyev. Lectures on Mathematical Aspects of Theoretical Physics at School of Mathematical Sciences: (Fall semester-2005).
- [4] B.Barnes, G.R.Fulford. Mathematical Modeling with Case Studies: Taylor and Francis Inc. (2002).
- [5] F.R.Giordano, M.D.Weir, William P.Fox. A First Course in Mathematical Modeling: GWO. (Second Edition) 1997.
- [6] G.Fulford, P.Forrester, A.Jones. Modeling with Differential and Difference Equations: Australian Mathematical Society Lecture Series 10. Cambridge University Press (1997).
- [7] S. V. Borisenok. Lectures on Differential equations at School of Mathematical Sciences: (Spring semester-2005).