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MODERN CONTROL THEORY AND IT'S APPLICATIONS

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Abstract

We discuss here the main concepts and some applications of modern control theory. We use the differential equation approach in the place of Laplace transform and show a simple example how control theory works in the case of modulated mechanical oscillator.

Introduction

Cybernetics and Control Theory

Originally this word "cybernetics" came from the Ancient Greece, and it meant "the art of steersman". Cybernetics is the "old-fashion" term for what is now called as the general system's theory, or the system science. We can admit that cybernetics was born in 1948, when the book with the same title has been written by the famous American mathematician Norbert Wiener (1894-1964) [1].

The dictionary of mathematics describes cybernetics as "the branch of science concerned with control systems… In a series of operations, information gained at one stage can be used to modify later performances of that operation. This is known as *feedback* and enables a control system to check and possibly adjust its actions when required" [2].

Practically some ideas of control were realized intuitively much more before of their precise mathematical formulations. The common example is Watt's regulator – a typical control mechanical system.

In [3] control theory is defined as "the mathematical study of how to manipulate the parameters affecting the behavior of a system to produce the desired or optimal outcome".

Even from these short definitions we can conclude, that such terms as controlled *dynamic system,* its *state*, *desired outcome* (also *goal* of the control in the modern language) and, sure, *feedback* are the main conceptions of control theory. Classical control theory is concentrated mostly on the problems of stability and stabilizability (in the forms of the Liapunov functions and other methods), as well as on the realization of the optimal control (see [4]). We, however, will omit this traditional approach and will discuss the aspects of the mathematical control, which are the most useful in the applications to the real problem of physics, both classical and quantum.

In the frame of control theory we can mark out two main different schools. In the USA and Western Europe the frequency domain (integral Laplace transform) was used primarily to describe the feedback in the terms of frequency variables. In contrast, in **Russia** and Eastern Europe a time-domain formulation using differential equations was dominated. To learn the control description in the form of Laplace transform we send our reader to [5-7]. Here we concentrate on the Russian school approach less known for Pakistani students.

1. Main definitions

In standard time-varying model we consider a nonlinear dynamic system of the form [8]:

$$\dot{x} = F(x,u,t)\,, \tag{1}$$

with the scalar *time t*, the *state* vector $x \in \mathrm{R}^n$ and the *controlling signal* vector $u \in \mathrm{R}^m$. Here F is a continuously differentiable vector function. For mechanical dynamic systems, for instance, the state space is their phase space. In the case of one-dimensional movement it concludes two elements of the state vector: spatial coordinate and velocity.

Sometimes, especially in engineering applications, we can define obviously the controlling part of the system (*controller*) and its controlled part (*plant*, or *process*) [5,6]. Practically, however, in a lot of modern applications (for instance, in quantum physics, in nanolithography [9]) it is very difficult to divide the system into a controller and a plant. For this reason we will use another control scheme.

Here for simplicity we will discuss the closed-loop scheme and, moreover, one-loop feedback only. A closed-loop control uses a measurement of the output and feedback of this signal to compare it with the desired meaning and to correct the system parameters. On Fig.1 we present the principal scheme of such a control.

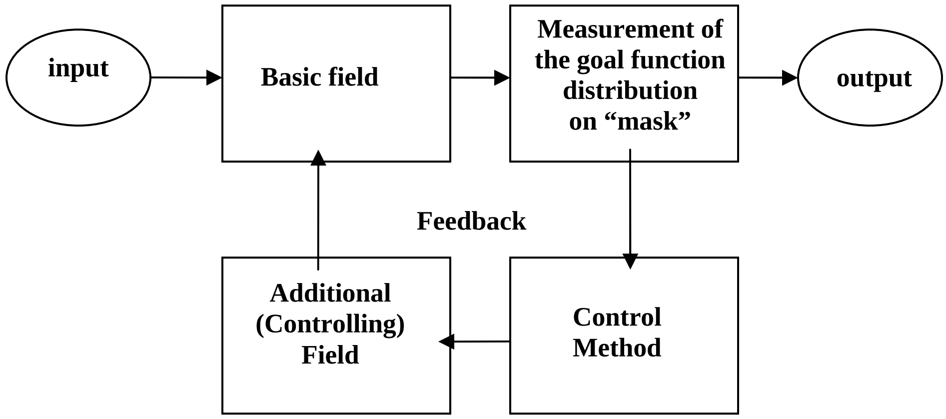

Fig. 1. Principal scheme of control feedback in dynamical system

The concrete realization of one-loop feedback scheme depends on two things: the first, of the precise mathematical definition of the control goal and, the second, of the expression for the feedback itself in the form of a functional dependency of the system state.

2. Goal functions and speed gradient control

The ***goal*** in the control process is a smooth scalar function Q with the limit relation

$$\lim_{t\to\infty} Q(x(t),t) \to 0 . \tag{2}$$

Very often the control goal should be expressed in the term of a physical energy. In order to ensure control the desired energy level H_*, the energy related goal function

$$Q = \left(H - H_*\right)^2 . \tag{3}$$

can be chosen, where H is the Hamiltonian function of the dynamical system. In this case we realize an *energy level control* in dynamic system[1].

[1] If the reader is not familiar with the Hamiltonian formulation of classical mechanics, he can understand the function H for simplicity as an ordinary E, i.e. the sum of kinetic and potential energies.

In (2) the goal function $Q(x(t),t)$ does not depend on $u(t)$. It means that it must be expressed through the controlling signal. For this reason let's define the speed of changing $Q(x(t),t)$ along the trajectories of (1), (i.e.) the time derivative $\omega(x,u,t) \equiv \dot{Q}$, then

$$\omega(x,u,t) = \frac{\partial Q(x,t)}{\partial t} + \left[\nabla_x Q(x,t)\right]^T \dot{x}$$

where T is transpose operation. Substituting of $\dot{x}$ from (1) we get:

$$\omega(x,u,t) = \frac{\partial Q(x,t)}{\partial t} + \left[\nabla_x Q(x,t)\right]^T F(x,u,t) \; . \qquad (4)$$

Here we will discuss one of the simplest, but in the same time very efficient methods of control, it is ***speed-gradient method***. In this approach the control action is chosen in the maximum descent direction for a scalar goal function. The input is expressed through some vector function ψ :

$$u = -\psi\left(\nabla_u \omega(x,u,t)\right). \qquad (5)$$

This vector $\psi(z)$ forms an acute angle with the vector *z*, (i. e) $\psi(z)^T z > 0$ when $z \neq 0$.

The control is often considered in the form of the proportional feedback:
$\psi(z) \equiv \kappa \cdot z$ with the constant $\kappa > 0$; or in the sign-proportional form:
$\psi(z) \equiv \kappa \cdot \text{sign}(z)$, $\kappa > 0$.

Now we have the *closed* system of differential equations (1)-(5) (and the closure came from the control theory; that is mathematical realization of feedback). If we know the physical parameters of a system and its Hamiltonian function (= its energy), then we can apply the speed-gradient methods to control the system behavior, say, to reach the desirable level of energy in the system.

3. Example: controlled Kapitza oscillator

One of the classical examples of the control theory is planar modulated oscillator (Kapitza pendulum[2]) [8]. The goal is to keep a mass attached to a rigid rod in some position, sometimes differ from the low horizontal one. The task can be achieved due to a many feedback methods, (i. e) by measurements of a temporary oscillator

[2] Peter Kapitza (another spelling is 'Piotr Kapitca') was a distinguish Russian physicist, the Nobel Prize laureate (1978).

position, and by introducing appropriate rod movement to balance the system. Simplest method is stabilization by *fast* vertical or/and horizontal vibrations of the pendulum support. Here for simplicity we will discuss only vertical type of oscillations.

Imitation modeling

$$ml^2 \cdot \ddot{\varphi} + \rho \cdot \dot{\varphi} + mgl \sin\varphi = mlu \sin\varphi \ , \tag{6}$$

where $\varphi = \varphi(t)$ is an angle (and the meaning $\kappa > 0$ corresponds to the low stable point), $u = u(t)$ is the vertical acceleration of the pendulum support, $\rho \geq 0$ is the friction coefficient. The vertical downward position $\varphi = 0$ is always stable. However, for the harmonic acceleration $u(t) = A\omega^2 \sin\omega t$ the vertical upward position $\varphi = \pi$ is also stable, if the condition holds:

$$\frac{a^2\omega^2}{2gl} > 1 \, . \tag{7}$$

We define the control goal as;

$$\lim_{t\to\infty} H(t) = H_* \ , \tag{8}$$

where

$$H(t) = \frac{ml^2}{2}(\dot{\varphi})^2 + mgl \cdot (1 - \cos\varphi) \tag{9}$$

is the full oscillator energy. Then in the frame of speed gradient method (3) we can write;

$$u(t) = -\gamma(H - H_*)\dot{\varphi} \cdot \sin\varphi \tag{10a}$$

$$u(t) = -\gamma \cdot \mathrm{sign}[(H - H_*)\dot{\varphi} \cdot \sin\varphi] \tag{10b}$$

The stabilization in the vertical upward position corresponds to

$$H_* = 2mgl \ . \tag{11}$$

Now we can investigate this system in the form of SIMULINK model (see Mat LAB). Our state space is two-dimensional: $x_1 = \varphi$ and $x_2 = \dot{\varphi}$.

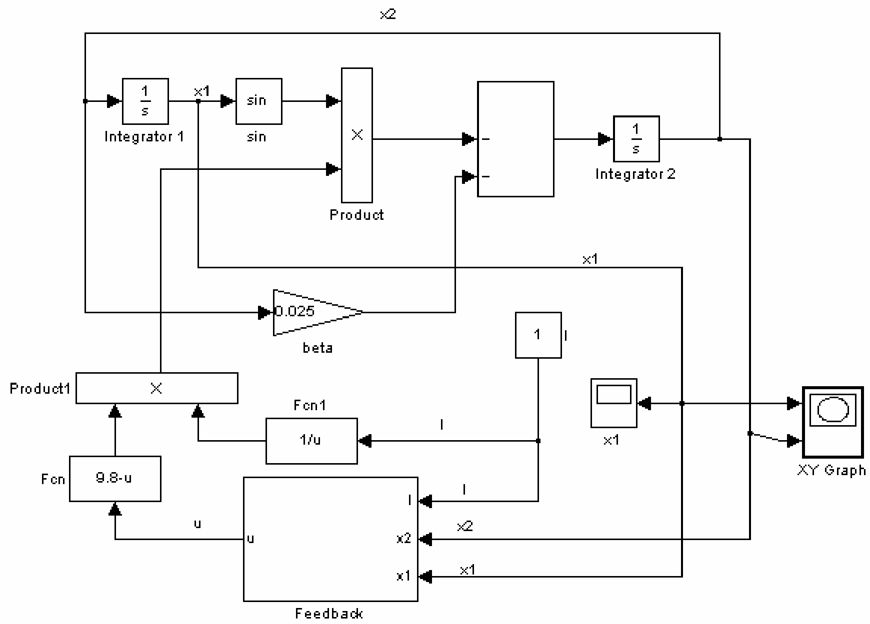

Fig. 2. SIMULINK model for the dynamical system (9). It presented in the form of the system (12).

On Fig. 2 we present the imitation of Eq. (6), which in our case should be written as a system:

$$\begin{cases} \dot{x}_1 = x_2; \\ \dot{x}_2 = -\beta \cdot x_2 - \dfrac{(g-u)}{l}\sin x_1. \end{cases} \tag{12}$$

The initial conditions for the state vector are included into the integrator blocks. (Can you recognize the feedback loop on the figure?) Here we use notation:

$$\beta = \rho / ml^2 .$$

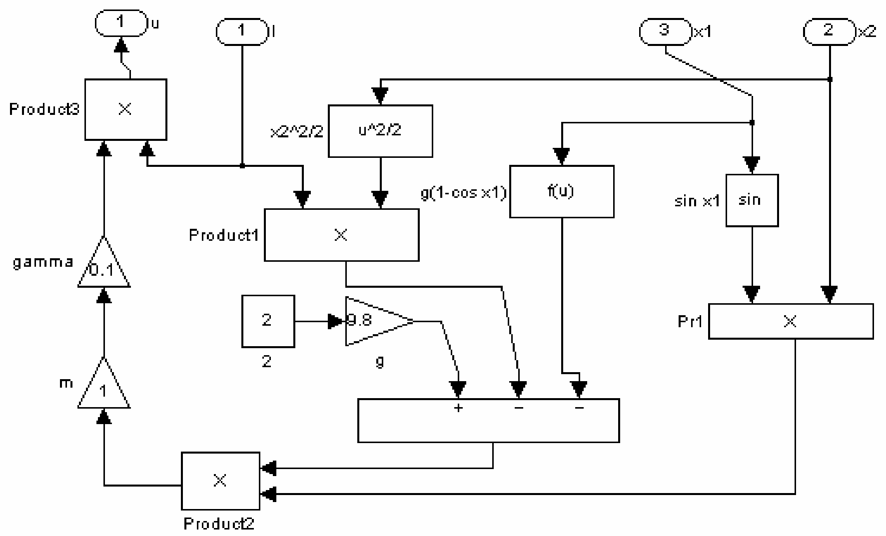

Fig. 3. SIMULINK model for the speed gradient control $u(x_1, x_2)$, corresponding to (10a). It is the internal part of the block "Feedback" on Fig.2

On Fig.3 the subsystem $u(x_1, x_2)$ is constructed, this "function" (in SIMULINK sense) corresponds to the speed gradient control scheme (10a) $u(x_1, x_2) = -\gamma (H - H_*) x_2 \sin x_1$ with the energy:

$$H = mgl \left[\frac{l}{2g} x_2^2 + 1 - \cos x_1 \right] . \quad (13)$$

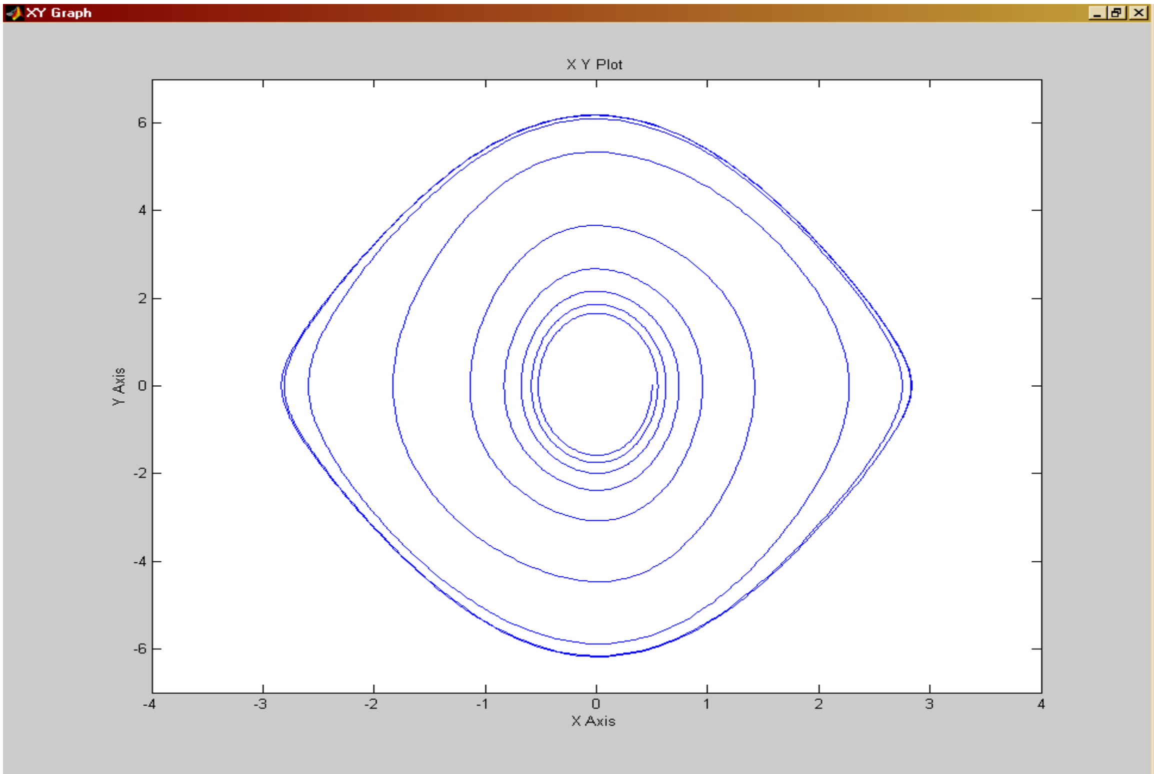

Fig. 4. XY-Graph for the speed gradient control simulation. *X*-axis corresponds to $x_1 = \varphi$, *Y*-axes corresponds to $x_2 = \dot{\varphi}$.

For practical calculations the meanings $l = 1$, $g = 9.8$, $\rho = 0.025$, $\gamma = 0.1$, and also $mgl \equiv 1$ have been chosen.

The results of our control you can see on Fig. 4. The system reaches the high level of energy (11) very rapidly.

Using SIMULINK, you can investigate the different properties of the controlled Kapitza pendulum.

4. Where it is possible to apply control theory?

Almost everywhere. From cybernetics and engineering devices the modern control theory spread out on the full variety of physical systems, it is applicable in mechanics, electrodynamics, optics, and in quantum mechanics. For instance, in some nano-lithography with the focusing of cooled atom beam the differential equations have almost one-to-one correspondence with the equation for mechanical Kapitza pendulum (with the more complicated "friction" term) [9].

5. Further reading on control theory

If you are interested to study its main principles and if you are more concentrated on the mathematical formulations, then [4] and [8]. If you want to know more about the applications, then [5-7] could be a good introduction. There are also a lot of materials in Internet, you can check, for example, [10] and other corresponding links. Anyway, control theory is a very rapidly developing science, still waiting for new energetic and talented young researchers.

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