

## ON SMOOTHING OF NON-SMOOTH FUNCTIONS

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ABSTRACT. A simple averaging method to smooth functions is offered. Application to investigation of underwater sound propagation is described.

### 1. Introduction

We describe in this note a method to smooth continuous non-smooth functions. When the original function is given by a table and it is necessary to restore the function in the whole interval, standard approximation methods such as polynomial or spline interpolation, least squares, and uniform approximations, are commonly used. However, in some special cases such methods can lead to undesirable effects. For instance, take a continuous function  $f(x)$ ,  $x \in [a, b]$ , choose node points  $x_1, \dots, x_n$  in  $[a, b]$  and find the polynomial to interpolate the table  $\{(x_j, f(x_j))\}$ ,  $j = 1, \dots, n$ . A table of a non-monotone function generates the interpolation polynomial which can have, in between the nodes, some "freak values" very different from the corresponding values of the original function  $f(x)$ .

The alternative idea is to approximate the function  $f(x)$  which is given by a table, with the help of a piece-wise linear function  $f_L(x)$  or, geometrically, to draw a linear broken line through the table points. Such approximation has the advantage that it does conserve the character of the original function and also because of simplicity of operations with linear functions in partial intervals between the adjacent nodes. The disadvantage is that the derivative  $f'_L(x)$  is discontinuous, and this discontinuity can strongly distort the result of computations in some important cases. So, it is desirable to get a method to construct a smooth function close to a piece-wise linear function. In the

next section we describe a smoothing method applicable to any continuous piece-wise smooth function.

## 2. The Smoothing Operator

Consider the class  $C^m$  of real functions  $f(x)$  continuous for  $x \in (-\infty, +\infty)$  and having continuous derivatives up to the  $m$ -th order,  $m \geq 0$ . We take a positive number  $\delta$  and introduce the *smoothing operator*  $S$  by the formula:

$$(Sf)(x) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(t) dt, \quad x \in (-\infty, +\infty). \quad (1)$$

We call the number  $\delta$  the *smoothing radius*. The transformation (1) is, in fact, the averaging over an interval of the length  $2\delta$ . For the transformed function,  $(Sf)(x)$ , we use also the short notation  $\tilde{f}(x)$ .

The smoothing operator  $S$  has the following properties:

P1.  $S$  is a linear operator:  $S(\alpha f(x) + \beta g(x)) = \alpha \tilde{f} + \beta \tilde{g}$ ,  $\alpha$  and  $\beta$  being constants;

P2. If  $f \in C^m$  then  $\tilde{f} \in C^{m+1}$ , i.e. the operator increases the smoothness of the function;

P3. If  $\delta \rightarrow 0$  then  $\tilde{f}(x) \rightarrow f(x)$ , so the smaller  $\delta$  the closer  $\tilde{f}$  to  $f$ ;

P4.  $S(\alpha x + \beta) = \alpha x + \beta$ , i.e. the linear function is invariant with respect to the operator;

P5. For  $n \geq 2$ ,  $S(x^n) = x^n + \frac{n(n-1)}{6} x^{n-2} \delta^2 + \dots + \frac{1}{2(n+1)} [\delta^n - (-\delta)^n]$ ;

It follows immediately from P5 that, for every polynomial  $P_n(x)$ , its transform  $(SP_n)(x)$  is also a polynomial  $\tilde{P}_n(x)$  of the same degree  $n$  and, moreover, the coefficient of the highest power of  $\tilde{P}_n(x)$  is the same as of  $P_n(x)$ .

The properties P1, P4, and P5 are verified by direct integration; the property P2 is established by taking the derivative of  $\tilde{f}(x)$ , and P3 with the help of l'Hospital rule.

## 3. The Smoothed Linear Broken Line

Our special interest is in the following: What is the transformed function in the case when the original one is a continuous piece-wise linear function? Or, geometrically, what type of the curve is the smoothed linear broken line?

Let the graph of the original function  $f(x)$  consists of two straight half-lines:  $y = ax + b$  for  $x \leq x_0$  and  $y = cx + d$  for  $x \geq x_0$ , where  $a \neq b$ . The two half-lines form the angle  $\Psi$ , see Figure No. 1.

Given a smoothing radius  $\delta > 0$ , suppose at first that  $x + \delta \leq x_0$ , or  $x \leq x_0 - \delta$ . Due to the property P4, it is true that  $\tilde{f}(x) = f(x)$ ; the same is true for  $x - \delta \geq x_0$ , or  $x \geq x_0 + \delta$ . We see that outside of the interval  $I := [x_0 - \delta, x_0 + \delta]$  nothing is changed and the lines remain to be straight.

Another situation is inside of this interval. Indeed, let  $x_0 - \delta < x < x_0 + \delta$ . The integral in  $S$  equals now the sum of two integrals:

$$\int_{x-\delta}^{x+\delta} f(t) dt = \int_{x-\delta}^{x_0} (at + b) dt + \int_{x_0}^{x+\delta} (ct + d) dt = Ax^2 + Bx + C,$$

where  $A = (c - a)/2$ ,  $B = (d - b) + \delta(a + c)$ ,  $C = x_0^2(a - c)/2 + x_0(b - d) + \delta(b + d) + \delta^2(c - a)/2$ . We see that the transformed function is a quadratic one, and its graph is a parabola. Since the original function is of class  $C^0$ , its transform is of class  $C^1$  (the property P2), which denotes geometrically that the parabola is tangent to both sides of the angle  $\Psi$ , see Figure No. 1.

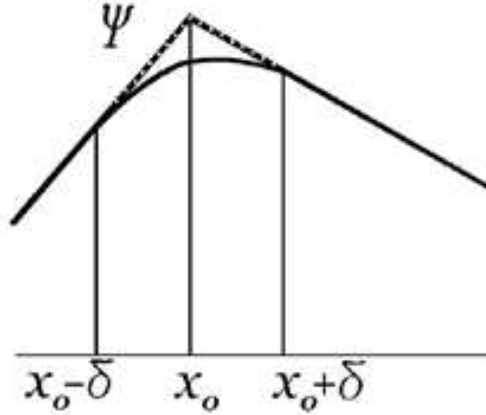


FIGURE 1. Broken line and smoothed broken line

In the general case, the original graph of  $f(x)$  is a broken line consisting of  $k \geq 2$  linear links  $\gamma_1, \gamma_2, \dots, \gamma_k$  and angle points at  $x = x_1, x_2, \dots, x_{k-1}$ . Choose the smoothing radius  $\delta < \frac{1}{2} \min_{2 \leq s \leq k} |x_s - x_{s-1}|$  and define the smoothing operator  $S$  by formula (1), then the transformed line,  $Sf(x)$ , is continuously differentiable. For small values of the smoothing radius  $\delta$ , the transformed

line is very close to the original one, and the difference can be made arbitrary small provided that  $\delta$  be taken sufficiently small.

#### 4. Application to the Phase Integral

In the theory of propagation of sounds in the sea or ocean water, some problems arise which require a proper restoration of functions given by a table. It should be noted that the investigation of underwater sound is of great theoretical and practical significance: such phenomena as earthquakes and tsunamis are related to powerful sound waves propagating in elastic media; underwater sound signals are used in searching of fish shoals and also by submarines. The complexity of such investigation is due to the fact that the actual water medium is always acoustically inhomogeneous: the sound speed  $v$  is different at different points, having average value about 1450 m/s; the value  $v$  depends on the depth, temperature and other factors. Because of this inhomogeneity, the sound waves propagate along complicated curve lines and can form in the sea water some zones of "loudness" and "silence": it is not unusual to hear the sound from the source at distance of hundreds kilometers (so-called waveguide propagation), and not hear it in several kilometers. Typical are the regions where the speed  $v$  of sound depends only on the vertical coordinate  $z$ ; such media are called layered or stratified. Often are met also regions where  $v$  depends strongly on  $z$  and weakly on other coordinates  $x, y$ .

Suppose the sound to be generated by a source at some depth under the sea surface and received by hydrophones (=underwater microphones) situated in water at some distance from the source. The desired value is the intensity of sound pressure as a function of distance (or time). Assume the sea water to be stratified,  $v = v(z)$ . One of popular methods for calculation of the wave field in such media is so-called *phase method*. The key role in this method plays the following function of the real parameter  $\mu > 0$ , called the *phase integral* [1]:

$$F(\mu) := \int_{\Delta(\mu)} \frac{dz}{\sqrt{n^2(z) - \mu^2}},$$

where  $n(z) = 1/v(z)$  and  $\Delta(\mu)$  is the interval (or intervals) such that  $\mu < n(z)$ . The integral is improper because the integrand is infinite at points where the value  $n^2(z) - \mu^2$  is vanishing. This circumstance makes difficult the application of standard numerical procedures for calculating the phase integral.

The function  $n^2(z)$  is usually given by a table  $\{(z_j, n_j^2)\}$ ,  $j = 1, \dots, N$  ( $N \sim 10 \div 100$ ), and it is necessary to restore  $n^2(z)$  as a continuous function in  $[z_1, z_N]$ . If the simplest restoration with the linear broken line is used, the

integral is decomposed into a sum of  $N - 1$  integrals taken along corresponding intervals between two neighboring nodes; each integral is an elementary function of the form

$$C_j \left( \sqrt{\alpha_j z_j + \beta_j - \mu^2} - \sqrt{\alpha_j z_{j-1} + \beta_j - \mu^2} \right).$$

The calculation is very simple, but it turns out that the function  $F(\mu)$  acquires "non-physical" angle points corresponding to the angle points of the broken line. As a result, the calculated wave field demonstrates corresponding dummy peaks, which have no physical sense.

Let us now smooth the linear broken-line restoration by the smoothing method described in the first section. In this way we get a smooth line  $\tilde{n}^2(z)$  which is close to the linear broken line provided the smoothing radius is taken small enough. Now the phase integral is decomposed into a sum of  $2N-1$  integrals, each of them can be calculated in terms of elementary functions such as  $\sqrt{\quad}$ ,  $\arcsin$ , and  $\ln$ . The calculation is also very effective, and the results are much more acceptable than those produced by the simplest broken-line restoration.

#### REFERENCES

- [1] Brekhovskikh, L.M. and Lysanov, Yu.P., *Fundamentals of ocean acoustics*, Springer-Verlag, Berlin, Germany, (1982).