

QUATERNION FORMS OF THE MOTIONS DEFINED AS $2 \times 3R$ ROBOTICS MODELLING

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ABSTRACT. A robot is a mechanical system which constitutes of pieces called link connected to the other through joints and of pieces which act in harmony with each other. The action of walking and climbing the stairs can be taken as a model of a robot to be defined. A special modelling that called $2 \times 3R$ robotics modelling has been given in [13] and [14]. The related robotic modelling consisting of triple revolute joints were defined and motion matrices of model and its parameters had been achieved. The inspiring source in $2 \times 3R$ robotic modelling was man's walking and climbing the stairs. The modelling of motion varies in every consecutive phases in terms of redescribing the fixed frame. Motion equations used in robotics modelling, sometimes, much more suitable in technical view. Form of matrices of motion was found in the kinematics analysis belong to $2 \times 3R$ robotics modelling which is defined at [13] and [14]. In this paper, equations of motion which is belong to $2 \times 3R$ robotics modelling has formulated by quaternion operator.

Key words : Step motion, transforming frame, kinematics, pole point, quaternion, climbing the stairs.

1. Introduction

1.1. **Displacements.** A mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a rigid transformation, if the following condition is satisfied :

$$\|X\| = \|F(X)\|$$

for all $X \in \mathbb{R}^n$, where, $\|\cdot\|$ is Euclidian norm in \mathbb{R}^n [1]. Rotations and translations in \mathbb{R}^n are rigid transformations and used commonly in kinematic studies [1]. To study the position of one body relative to another, we attach coordinate frames to each. One is chosen as the ground with coordinate frame F , and the other, the moving body, has the coordinate frame M . We use the coordinate transformation

$$D : F \rightarrow M$$

which transforms coordinates measured in M to those measured in F , to represent the position of M relative to F . This transformation is given by

$$X = [A]x + d$$

where x is the coordinate vector of a point in M and X is coordinate vector of the same point but measured in F . If the moving body is of dimension n (usually $n=2$ or 3), then $[A]$ is an $n \times n$ matrix and d is an n - dimensional vector. This transformation's matrix form is

$$D = \begin{bmatrix} A & d \\ 0 & 1 \end{bmatrix}$$

[7]. Motion which is given by matrix D can be given also with quaternion.

1.2. Quaternion Algebra. In 1843 Hamilton invented the so-called hyper-complex number of rank 4, to which he gave the name *quaternion*. Crucial to this invention was his celebrated rule

$$i^2 = j^2 = k^2 = -1$$

for dealing with the operations on the vector part of the quaternion. In particular we use \mathbf{i}, \mathbf{j} and \mathbf{k} to denote the standard orthonormal basis for R^3 . Now a quaternion, as the name already suggests, may be regarded as a 4-tuple of real numbers, that is, as an element of R^4 . In this case we would write

$$q = (q_0, q_1, q_2, q_3)$$

where q_0, q_1, q_2 and q_3 are simply real numbers or scalars. We shall adopt an alternative way of representing a quaternion. First, we define a *scalar part* to be some real number or scalar, say q_0 . Then we define a *vector part*, say \mathbf{q} , which is an ordinary vector in R^3

$$\mathbf{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

where \mathbf{i}, \mathbf{j} and \mathbf{k} are the standard orthonormal basis in R^3 . We now define a *quaternion* as the sum

$$Q = q_0 + \mathbf{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3.$$

In this sum, q_0 is called the *scalar part* of the quaternion while \mathbf{q} is called the *vector part* of the quaternion. The scalars q_0, q_1, q_2, q_3 are called the *components* of the quaternion.

Equality and Addition: We begin by saying two quaternions are *equal* if and only if they have exactly the same components, this is to say that if

$$P = p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3$$

and

$$Q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

then $P = Q$ if and only if

$$\begin{aligned} p_0 &= q_0 \\ p_1 &= q_1 \\ p_2 &= q_2 \\ p_3 &= q_3. \end{aligned}$$

The sum of the two quaternions P and Q above is defined by adding the corresponding components, that is

$$P + Q = (p_0 + q_0) + \mathbf{i}(p_1 + q_1) + \mathbf{j}(p_2 + q_2) + \mathbf{k}(p_3 + q_3).$$

Multiplication: The product of two quaternions $P = p_0 + \mathbf{p}$ and $Q = q_0 + \mathbf{q}$ in the more concise form is

$$P \otimes Q = p_0q_0 - \langle \mathbf{p}, \mathbf{q} \rangle + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}.$$

where $\langle \cdot, \cdot \rangle$ is the (Euclidian) inner product and \times is the vectorial product in \mathbf{R}^3 [3].

The Complex Conjugate: The *complex conjugate* of the quaternion

$$Q = q_0 + \mathbf{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

to be the quaternion, denoted by Q^* , and given by

$$Q^* = q_0 - \mathbf{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3.$$

The Norm: The norm of a quaternion Q , denoted by $N(Q)$ or $|Q|$, sometimes called the length of Q , is the scalar defined

$$N(Q) = \sqrt{Q^*Q}.$$

Inverse Of The Quaternion: Inverse of the quaternion is denoted by Q^{-1} and is defined

$$Q^{-1} = \frac{Q^*}{N^2(Q)}.$$

Quaternion Description Of Rotations: It is well known that rotation matrices $X' = R(\theta, n)X$ correspondence with the unit quaternions, Q , where $|Q| = 1$

$$X' = QXQ^*.$$

The scalar part, x_0 , of X is transformed into $x'_0 = (QQ^*)x_0 = x_0$. $X'.X' = X.X$. Since the scalar part is left unchanged, we also have $\mathbf{x}'.\mathbf{x}' = \mathbf{x}.\mathbf{x}$, which is sufficient to prove that the transformation is a rotation. The vector part, \mathbf{x} of \mathbf{X} is transformed into

$$\mathbf{x}' = (q_0^2 - \mathbf{q}.\mathbf{q})\mathbf{x} + 2q_0\mathbf{q} \times \mathbf{x} + 2\mathbf{q}(\mathbf{q}.\mathbf{x}). \quad (1)$$

It is reminiscent of the trigonometric double angle formulas. Suppose we want to rotate by θ about a line passing through the origin, and pointing in the direction \mathbf{n} . The theorem in the last section shows that we can accomplish this transformation maps vector to the vector

$$v' = n_2n_1vn_1n_2.$$

This can be written as

$$v' = C_q(v) = qvq^{-1}$$

where

$$q = n_2n_1.$$

In order to do this we have used the fact that since n_i is associated with a unit vector, then $n_i^{-1} = -n_i$, and also the fact that $(n_2n_1)^{-1} = n_1^{-1}n_2^{-1}$. A direct calculation shows that

$$Q = \pm(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})\mathbf{n}) \quad (2)$$

where \mathbf{n} is a unit vector and the sign depends on the choice of unit normals for the two planes. Clearly, the rotation is independent of the sign of Q . By choosing

$$Q = Q(\theta, \mathbf{n}) = \cos \frac{\theta}{2} + \sin \frac{\theta}{2}\mathbf{n}, \quad (3)$$

we make the two equations identical. This can also be written in a form similar to $\mathbf{R}(\theta, \mathbf{n}) = \exp(\theta\mathbf{N}) = e^{\theta\mathbf{N}}$. First, note that, as a *quaternion product*, $\mathbf{n}^2 = -1$. By manipulation of the power series, just as was done for rotation matrices, we find

$$Q(\theta, \mathbf{n}) = e^{(\frac{\theta}{2})\mathbf{n}} = \cos \frac{\theta}{2} + \sin \frac{\theta}{2}\mathbf{n}. \quad (4)$$

We have now exhibited the correspondence between unit quaternions and proper orthogonal matrices as representations of rotations.

$$Q(\theta, \mathbf{n}) \longleftrightarrow \mathbf{R}(\theta, \mathbf{n}).$$

This correspondence is a group homomorphism because $P(QXQ^*)P^* = PQXQ^*P^* = (PQ)X(PQ)^*$.

It is not quite an isomorphism, since both $Q(\theta, \mathbf{n})$ and $-Q(\theta, \mathbf{n})$ correspond to the same orthogonal matrix. Given the elements of a unit quaternion, Q , the elements of the corresponding rotation matrix, \mathbf{R} is obtained,

$$\mathbf{R} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(-q_0q_3 + q_1q_2) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_2q_1) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(-q_0q_1 + q_2q_3) \\ 2(-q_0q_2 + q_3q_1) & 2(q_0q_1 + q_3q_2) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}.$$

Dual Quaternion Kinematics Equations: The kinematics equations of the serial chain can be defined using elements of the clifford algebra, known as *dual quaternions*, instead of 4×4 homogenous transforms. The advantage is primarily a compact representation of the rotation matrix, and also a useful structure that assists the elimination of the joint variables. A spatial displacement consisting of a rotation by θ and slide by d around and along a screw S is written as the dual quaternion

$$\widehat{S} = \sin\left(\frac{\widehat{\theta}}{2}\right)S + \cos\left(\frac{\widehat{\theta}}{2}\right),$$

where $\widehat{\theta} = \theta + \epsilon d$ and $S = \mathbf{S} + \epsilon p \times \mathbf{S}$ is the dual vector formed from the Plücker coordinates of the screw axis. Recall that ϵ is the dual unit with the property $\epsilon^2 = 0$, and that the sine and cosine of a dual angle are defined by

$$\cos \frac{\widehat{\theta}}{2} = \cos \frac{\theta}{2} - \epsilon \frac{d}{2} \sin \frac{\theta}{2},$$

and

$$\sin \frac{\widehat{\theta}}{2} = \sin \frac{\theta}{2} + \epsilon \frac{d}{2} \cos \frac{\theta}{2}.$$

1.3. Dual Quaternions and Spatial Displacements. The coordinate transformation, $[T] = [A, \mathbf{d}]$, that defines the position of a body M relative to fixed frame F can be represented by a dual quaternion $\widehat{Z} = Z + \epsilon Z^0$. The real part, $Z = Z_4 + Z_1i + Z_2j + Z_3k$, is defined by the Euler parameters of the rotation $[A]$. The dual part, Z^0 is given by the formula:

$$Z^0 = \left(\frac{1}{2}\right)DZ$$

where $D = d_1i + d_2j + d_3k$ is the quaternion formed from the translation vector $\mathbf{d} = (d_1, d_2, d_3)$. Let \mathbf{d} be written in terms of the screw parameters of the displacement,

$\mathbf{d} = ds + c - [A]c$, given by $\mathbf{d} = ds + [I - A]c$. Writing each of these vectors in quaternion form we obtain

$$D = dS + C - ZCZ^*,$$

and

$$Z^0 = \left(\frac{1}{2}\right)(dSZ + CZ - ZC).$$

Expanding this expression we obtain

$$\begin{aligned} Z_1^0 &= \left(\frac{d}{2}\right)\cos\left(\frac{\phi}{2}\right)s_x + \sin\left(\frac{\phi}{2}\right)s_x^*, \\ Z_2^0 &= \left(\frac{d}{2}\right)\cos\left(\frac{\phi}{2}\right)s_y + \sin\left(\frac{\phi}{2}\right)s_y^*, \\ Z_3^0 &= \left(\frac{d}{2}\right)\cos\left(\frac{\phi}{2}\right)s_z + \sin\left(\frac{\phi}{2}\right)s_z^*, \\ Z_4^0 &= -\left(\frac{d}{2}\right)\sin\left(\frac{\phi}{2}\right) \end{aligned}$$

where $s^* = c \times s$. Let the dual vector $\widehat{s} = s + \epsilon c \times s$ represent the screw axis and let $\widehat{\phi} = \phi + \epsilon d$ be the dual angle defining the rotation about and translation along \widehat{s} , then we find that the dual quaternion $\widehat{Z} = Z + \epsilon Z^0$ becomes

$$\widehat{Z} = \cos\left(\frac{\widehat{\phi}}{2}\right) + \widehat{s}_x \sin\left(\frac{\widehat{\phi}}{2}\right)i + \widehat{s}_y \sin\left(\frac{\widehat{\phi}}{2}\right)j + \widehat{s}_z \sin\left(\frac{\widehat{\phi}}{2}\right)k.$$

Thus the components of a dual quaternion are obtained by replacing the Euler parameters with their dual versions, known as the *dual Euler parameters* of the spatial displacement. Using the dual Euler parameters, we can represent the dual orthogonal matrix, $[\widehat{A}] = [A] + \epsilon[D][A]$, by a dual version of $[A] = I + 2\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)[S] + 2\sin^2\left(\frac{\phi}{2}\right)[S^2]$:

$$[\widehat{A}] = I + 2\sin\left(\frac{\widehat{\phi}}{2}\right)\cos\left(\frac{\widehat{\phi}}{2}\right)[\widehat{S}] + 2\sin^2\left(\frac{\widehat{\phi}}{2}\right)[\widehat{S}^2].$$

Let $\widehat{\mathbf{w}} = \mathbf{w} + \epsilon \mathbf{v}$ be the dual vector form of the screw $\mathbf{W} = (w, v)$. This dual vector can be identified with the dual quaternion

$$\widehat{\mathbf{w}} = (w_1 + \epsilon v_1)i + (w_2 + \epsilon v_2)j + (w_3 + \epsilon v_3)k$$

called a *dual vector quaternion*. If $\widehat{\mathbf{w}}$ defines the coordinates of a screw in reference frame M , then the coordinates of the screw measured in another frame F is given by the dual quaternion $\widehat{\mathbf{w}}'$, defined by the transformation:

$$\widehat{\mathbf{w}}' = \widehat{Z}\widehat{\mathbf{w}}\widehat{Z}^*$$

where \widehat{Z}^* is the conjugate of \widehat{Z} [6]. For some related papers or books on the topics, one can also refer to the works [4], [5], [8], [9], [10], [12], [15], [11] and the references mentioned therein.

1.4. Modelling a Robot: Link Coordinate System Assignment. Given an n degree of freedom robot arm, this algorithm assigns an orthonormal coordinate system to each link of the robot arm according to arm configurations similar to those of human arm geometry. The labeling of the coordinate systems begins from the supporting base to the end-effector of the robot arm. The relations between adjacent links can be represented by a 4×4 homogeneous transformation matrix.

D1. *Establish the base coordinate system.* Establish a right-handed orthonormal coordinate system $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ at the supporting base with the \mathbf{z}_0 axis lying along the axis of motion of joint 1 and pointing toward the shoulder of the robot arm. The \mathbf{x}_0 and \mathbf{y}_0 axes can be conveniently established and are normal to the \mathbf{z}_0 axis.

D2. *Initialize and loop.* For each $i, i = 1, \dots, n - 1$, perform steps D3 to D6.

D3. *Establish joint axis.* Align the \mathbf{z}_i with the axis of motion (rotary and sliding) of joint $i + 1$. For robots having left-right arm configurations, the \mathbf{z}_1 and \mathbf{z}_2 axes are pointing away from the shoulder and the "trunk" of the robot arm.

D4. *Establish the origin of the i th coordinate system.* Locate the origin of the i th coordinate system at the intersection of the \mathbf{z}_i and \mathbf{z}_{i-1} axes or at the intersection of common normal between the \mathbf{z}_i and \mathbf{z}_{i-1} axes and the \mathbf{z}_i axis.

D5. *Establish \mathbf{x}_i axis.* Establish $\mathbf{x}_i = \pm \frac{(\mathbf{z}_{i-1} \times \mathbf{z}_i)}{\|\mathbf{z}_{i-1} \times \mathbf{z}_i\|}$ or along the common normal between the \mathbf{z}_{i-1} and \mathbf{z}_i axes when they are parallel.

D6. *Establish \mathbf{y}_i axis.* Assign $\mathbf{y}_i = + \frac{(\mathbf{z}_i \times \mathbf{x}_i)}{\|\mathbf{z}_i \times \mathbf{x}_i\|}$ to complete the right-handed coordinate system.

D7. *Establish the hand coordinate system.* Usually the n th joint is a rotary joint. Establish \mathbf{z}_n along the direction of \mathbf{z}_{n-1} axis and pointing away from the robot. Establish \mathbf{x}_n such that it is normal to both \mathbf{z}_{n-1} and \mathbf{z}_n axes. Assign \mathbf{y}_n to complete the right-handed coordinate system.

D8. *Find joint and link parameters.* For each $i, i = 1, \dots, n - 1$, perform steps D9 to D12.

D9. *Find d_i ;* d_i is the distance from the origin of the $(i - 1)$ th coordinate system to the intersection of the \mathbf{z}_{i-1} axis and the \mathbf{x}_i axis along the \mathbf{z}_{i-1} axis. It is the joint variable if joint i is prismatic.

D10. *Find a_i ;* a_i is the distance from the intersection of the \mathbf{z}_{i-1} axis and the \mathbf{x}_i axis to the origin of the i th coordinate system along the \mathbf{x}_i axis.

D11. *Find θ_i ;* θ_i is the angle of rotation from the \mathbf{x}_{i-1} axis to the \mathbf{x}_i axis about the \mathbf{z}_{i-1} axis. It is the joint variable if joint i is rotary.

D12. *Find α_i ;* α_i is the angle of rotation from the \mathbf{z}_{i-1} axis to the \mathbf{z}_i axis about the \mathbf{x}_i axis [2].

2. Modelling of $2 \times 3R$

In this section we will give a short summary from [13] and [14] which include some necessary knowledge that are used in Section 3. At the starting stage, the model parts are as in Figure 1. Let's take 2- models, which are represented by Robot L and Robot R , with 3 revolute joint. Model which consists of two parts is: Part $L = \{J_{00}, J_{01}, J_{02}, J_{03}\}$ and Part $R = \{J_{06}, J_{05}, J_{04}, J_{03}\}$. We defined $2 \times 3R$ robotics modelling as a model which consists of Part L and Part R .

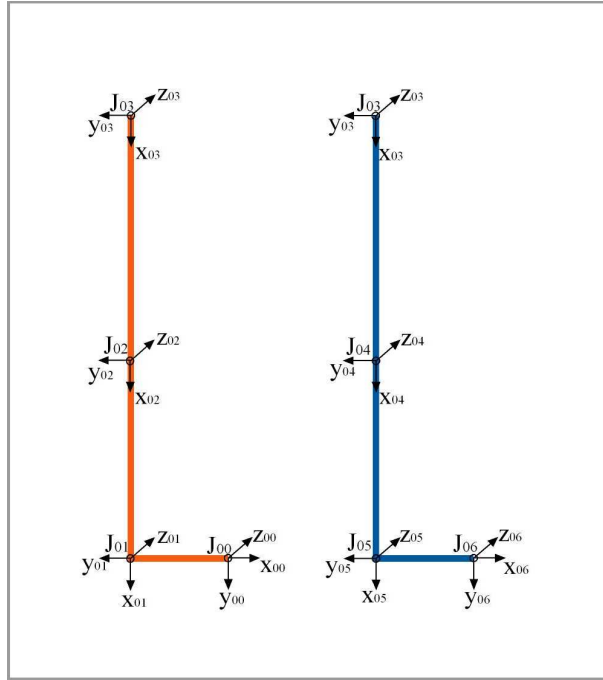


FIGURE 1

2.1. Matrices of the Motion. In order to be able to write the matrices of the motion, initially, let's construct a mechanical system and replace a frame to every joint:

The frame at J_{00} is F_{00} (fixed frame), the frame at J_{01} is M_{01} (moving frame), the frame at J_{02} is M_{02} (moving frame), the frame at J_{03} is M_{03} (moving frame), the frame at J_{04} is M_{04} (moving frame), the frame at J_{05} is M_{05} (moving frame), the frame at J_{06} is M_{06} (moving frame). This mechanism is like Figure 1. The diagram of the first step has been given in Figure 2.

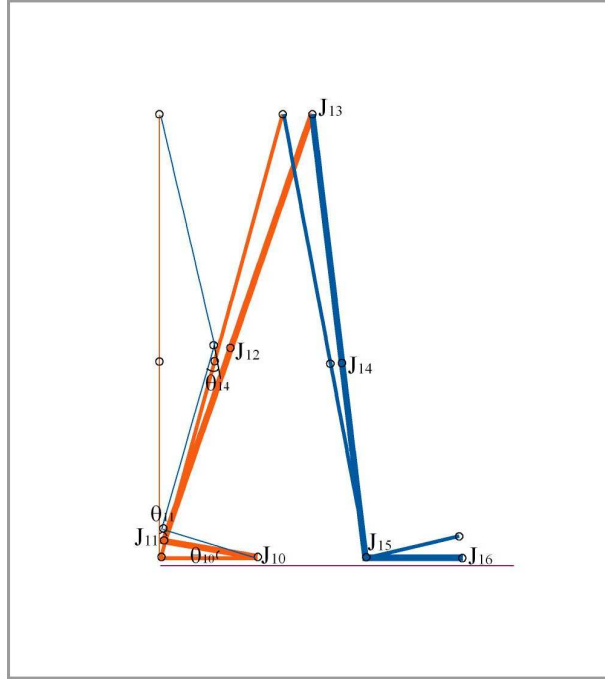


FIGURE 2

Let's show the first leg motion with W_1 . Matrix of the motion which takes $P_1(cC\theta, cS\theta)$ as pole point:

$$[W_{1p_1}] = \begin{bmatrix} C\theta_{14} & -S\theta_{14} & 0 & cC\theta - cC(\theta + \theta_{14}) \\ S\theta_{14} & C\theta_{14} & 0 & cS\theta - cS(\theta + \theta_{14}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

matrix of the motion which takes $P_2(b + c, 0)$ as pole point:

$$[W_{1p_2}] = \begin{bmatrix} C\theta_{11} & S\theta_{11} & 0 & (b + c)(1 - C\theta_{11}) \\ -S\theta_{11} & C\theta_{11} & 0 & (b + c)S\theta_{11} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

matrix of the motion which takes $P_3(b + c, a)$ as pole point:

$$[W_{1p_3}] = \begin{bmatrix} C\theta_{10} & S\theta_{10} & 0 & (b + c)(1 - C\theta_{10}) - aS\theta_{10} \\ -S\theta_{10} & C\theta_{10} & 0 & (b + c)S\theta_{10} + a(1 - C\theta_{10}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Similarly, the matrix of ${}_2W$ is

$$[{}_2W] = [{}_{3p}{}_2W][{}_{2p}{}_2W][{}_{1p}{}_2W],$$

where,

$$[{}_{1p}{}_2W] = \begin{bmatrix} C_{25}\theta & S_{25}\theta & 0 & d_1 \\ -S_{25}\theta & C_{25}\theta & 0 & d_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[{}_{2p}{}_2W] = \begin{bmatrix} C_{23}\theta & -S_{23}\theta & 0 & d'_1 \\ S_{23}\theta & C_{23}\theta & 0 & d'_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[{}_{3p}{}_2W] = \begin{bmatrix} C_{26}\theta & S_{26}\theta & 0 & d''_1 \\ -S_{26}\theta & C_{26}\theta & 0 & d''_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix of the third step is

$$[W_3] = [W_{3p3}][W_{3p2}][W_{3p1}],$$

where,

$$[W_{3p1}] = \begin{bmatrix} C\theta_{31} & S\theta_{31} & 0 & d_1 \\ -S\theta_{31} & C\theta_{31} & 0 & d_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[W_{3p2}] = \begin{bmatrix} C\theta_{33} & -S\theta_{33} & 0 & d'_1 \\ S\theta_{33} & C\theta_{33} & 0 & d'_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[W_{3p3}] = \begin{bmatrix} C\theta_{30} & S\theta_{30} & 0 & d''_1 \\ -S\theta_{30} & C\theta_{30} & 0 & d''_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix of WS is

$$[WS] = [{}_{3p}WS][{}_{2p}WS][{}_{1p}WS],$$

where,

$$[{}_{1p}WS] = \begin{bmatrix} C_{(n+1)5}\theta & S_{(n+1)5}\theta & 0 & d_1 \\ -S_{(n+1)5}\theta & C_{(n+1)5}\theta & 0 & d_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[{}_{2p}WS] = \begin{bmatrix} C_{(n+1)3}\theta & -S_{(n+1)3}\theta & 0 & d'_1 \\ S_{(n+1)3}\theta & C_{(n+1)3}\theta & 0 & d'_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[{}_{3p}WS] = \begin{bmatrix} C_{(n+1)2}\theta & S_{(n+1)2}\theta & 0 & d''_1 \\ -S_{(n+1)2}\theta & C_{(n+1)2}\theta & 0 & d''_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix of S_1 is

$$[S_1] = [S_{1p_5}] [S_{1p_4}] [S_{1p_3}] [S_{1p_2}] [S_{1p_1}],$$

where,

$$[S_{1p_1}] = \begin{bmatrix} C\theta_{(n+2)1} & S\theta_{(n+2)1} & 0 & d_1 \\ -S\theta_{(n+2)1} & C\theta_{(n+2)1} & 0 & d_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[S_{1p_2}] = \begin{bmatrix} C\theta_{(n+2)1} & -S\theta_{(n+2)1} & 0 & d'_1 \\ S\theta_{(n+2)1} & C\theta_{(n+2)1} & 0 & d'_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[S_{1p_3}] = \begin{bmatrix} C\theta_{(n+2)2} & S\theta_{(n+2)2} & 0 & d''_1 \\ -S\theta_{(n+2)2} & C\theta_{(n+2)2} & 0 & d''_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[S_{1p_4}] = \begin{bmatrix} C\theta_{(n+2)3} & -S\theta_{(n+2)3} & 0 & d'''_1 \\ S\theta_{(n+2)3} & C\theta_{(n+2)3} & 0 & d'''_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[S_{1p_5}] = \begin{bmatrix} C\theta_{(n+2)4} & S\theta_{(n+2)4} & 0 & d''''_1 \\ -S\theta_{(n+2)4} & C\theta_{(n+2)4} & 0 & d''''_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix of ${}_2S$ is

$$[{}_2S] = [{}_{5p}{}_2S] [{}_{4p}{}_2S] [{}_{3p}{}_2S] [{}_{2p}{}_2S] [{}_{1p}{}_2S],$$

where,

$$\begin{aligned}
 [{}_{1p}2S] &= \begin{bmatrix} C_{(n+3)5\theta} & S_{(n+3)5\theta} & 0 & d_1 \\ -S_{(n+3)5\theta} & C_{(n+3)5\theta} & 0 & d_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
 [{}_{2p}2S] &= \begin{bmatrix} C_{(n+3)5\theta} & -S_{(n+3)5\theta} & 0 & d'_1 \\ S_{(n+3)5\theta} & C_{(n+3)5\theta} & 0 & d'_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
 [{}_{3p}2S] &= \begin{bmatrix} C_{(n+3)4\theta} & S_{(n+3)4\theta} & 0 & d''_1 \\ -S_{(n+3)4\theta} & C_{(n+3)4\theta} & 0 & d''_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
 [{}_{4p}2S] &= \begin{bmatrix} C_{(n+3)3\theta} & -S_{(n+3)3\theta} & 0 & d'''_1 \\ S_{(n+3)3\theta} & C_{(n+3)3\theta} & 0 & d'''_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
 [{}_{5p}2S] &= \begin{bmatrix} C_{(n+3)2\theta} & S_{(n+3)2\theta} & 0 & d''''_1 \\ -S_{(n+3)2\theta} & C_{(n+3)2\theta} & 0 & d''''_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
 \end{aligned}$$

The matrix of SW is

$$[SW] = [SW_{p_5}][SW_{p_4}][SW_{p_3}][SW_{p_2}][SW_{p_1}],$$

where,

$$\begin{aligned}
 [SW_{p_1}] &= \begin{bmatrix} C\theta_{(n+m+1)1} & S\theta_{(n+m+1)1} & 0 & d_1 \\ -S\theta_{(n+m+1)1} & C\theta_{(n+m+1)1} & 0 & d_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
 [SW_{p_2}] &= \begin{bmatrix} C\theta_{(n+m+1)1} & -S\theta_{(n+m+1)1} & 0 & d'_1 \\ S\theta_{(n+m+1)1} & C\theta_{(n+m+1)1} & 0 & d'_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
 [SW_{p_3}] &= \begin{bmatrix} C\theta_{(n+m+1)2} & S\theta_{(n+m+1)2} & 0 & d''_1 \\ -S\theta_{(n+m+1)2} & C\theta_{(n+m+1)2} & 0 & d''_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
 \end{aligned}$$

$$[SW_{p_4}] = \begin{bmatrix} C\theta_{(n+m+1)3} & -S\theta_{(n+m+1)3} & 0 & d_1''' \\ S\theta_{(n+m+1)3} & C\theta_{(n+m+1)3} & 0 & d_2''' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[SW_{p_5}] = \begin{bmatrix} C\theta_{(n+m+1)0} & S\theta_{(n+m+1)0} & 0 & d_1'''' \\ -S\theta_{(n+m+1)0} & C\theta_{(n+m+1)0} & 0 & d_2'''' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Quaternion Forms of the Motions Defined as $2 \times 3R$ Robotics Modelling

The matrix of W_1 was

$$[W_1] = [W_{1p_3}][W_{1p_2}][W_{1p_1}].$$

The rotation axes and the rotation angles which are belong to parts of motion of W_1 are given at Table 1.

	Axis	Rotation angle	Quaternion form
$[W_{1p_1}]$	$z - axis$	$\theta_{14}(t_{11})$	$Q_{W_{1p_1}}$
$[W_{1p_2}]$	$z - axis$	$\theta_{11}(t_{12})$	$Q_{W_{1p_2}}$
$[W_{1p_3}]$	$z - axis$	$\theta_{10}(t_{13})$	$Q_{W_{1p_3}}$

Table 1

And, $Q_{W_{1p_1}}, Q_{W_{1p_2}}, Q_{W_{1p_3}}$ quaternion forms are as follows:

$$\begin{aligned} Q_{W_{1p_1}} &= \left(C\frac{\theta_{14}(t_{11})}{2} + (0, 0, 1)S\frac{\theta_{14}(t_{11})}{2} \right), \\ Q_{W_{1p_2}} &= \left(C\frac{\theta_{11}(t_{12})}{2} - (0, 0, 1)S\frac{\theta_{11}(t_{12})}{2} \right), \\ Q_{W_{1p_3}} &= \left(C\frac{\theta_{10}(t_{13})}{2} - (0, 0, 1)S\frac{\theta_{10}(t_{13})}{2} \right). \end{aligned}$$

Finally, quaternion form of W_1 is

$$\begin{aligned}
 Q_{W_1} &= Q_{W_{1p_3}} \otimes Q_{W_{1p_2}} \otimes Q_{W_{1p_1}} \\
 &= \left(C \frac{\theta_{10}(t_{13})}{2} - (0, 0, 1) S \frac{\theta_{10}(t_{13})}{2} \right) \otimes \left(C \frac{\theta_{11}(t_{12})}{2} - (0, 0, 1) S \frac{\theta_{11}(t_{12})}{2} \right) \\
 &\quad \otimes \left(C \frac{\theta_{14}(t_{11})}{2} + (0, 0, 1) S \frac{\theta_{14}(t_{11})}{2} \right) \\
 &= \left(C \frac{\theta_{10}(t_{13})}{2} - (0, 0, S \frac{\theta_{10}(t_{13})}{2}) \right) \otimes \left(C \frac{\theta_{11}(t_{12})}{2} - (0, 0, S \frac{\theta_{11}(t_{12})}{2}) \right) \\
 &\quad \otimes \left(C \frac{\theta_{14}(t_{11})}{2} + (0, 0, S \frac{\theta_{14}(t_{11})}{2}) \right) \\
 &= \left(\left(C \frac{\theta_{10}(t_{13})}{2} C \frac{\theta_{11}(t_{12})}{2} - S \frac{\theta_{10}(t_{13})}{2} S \frac{\theta_{11}(t_{12})}{2} \right) \right. \\
 &\quad \left. + (0, 0, -C \frac{\theta_{10}(t_{13})}{2} S \frac{\theta_{11}(t_{12})}{2} - S \frac{\theta_{10}(t_{13})}{2} C \frac{\theta_{11}(t_{12})}{2}) \right) \\
 &\quad \otimes \left(C \frac{\theta_{14}(t_{11})}{2} + (0, 0, S \frac{\theta_{14}(t_{11})}{2}) \right) \\
 &= \left(C \frac{\theta_{10}(t_{13}) + \theta_{11}(t_{12})}{2} + (0, 0, -S \frac{\theta_{10}(t_{13}) + \theta_{11}(t_{12})}{2}) \right) \\
 &\quad \otimes \left(C \frac{\theta_{14}(t_{11})}{2} + (0, 0, S \frac{\theta_{14}(t_{11})}{2}) \right) \\
 &= \left(C \frac{\theta_{10}(t_{13}) + \theta_{11}(t_{12})}{2} C \frac{\theta_{14}(t_{11})}{2} + S \frac{\theta_{10}(t_{13}) + \theta_{11}(t_{12})}{2} S \frac{\theta_{14}(t_{11})}{2} \right) \\
 &\quad + (0, 0, C \frac{\theta_{10}(t_{13}) + \theta_{11}(t_{12})}{2} S \frac{\theta_{14}(t_{11})}{2} - S \frac{\theta_{10}(t_{13}) + \theta_{11}(t_{12})}{2} C \frac{\theta_{14}(t_{11})}{2}) \\
 &= \left(C \frac{\theta_{14}(t_{11}) - \theta_{10}(t_{13}) - \theta_{11}(t_{12})}{2} + (0, 0, S \frac{\theta_{14}(t_{11}) - \theta_{10}(t_{13}) - \theta_{11}(t_{12})}{2}) \right) \\
 &= \left(C \frac{\theta_{14}(t_{11}) - \theta_{10}(t_{13}) - \theta_{11}(t_{12})}{2} + (0, 0, 1) S \frac{\theta_{14}(t_{11}) - \theta_{10}(t_{13}) - \theta_{11}(t_{12})}{2} \right).
 \end{aligned}$$

Similarly, we can write quaternion form of ${}_2W$ and W_3 as follows;

$$\begin{aligned}
 Q_{{}_2W} &= Q_{{}_3p_2W} \otimes Q_{{}_2p_2W} \otimes Q_{{}_1p_2W} \\
 &= \left(C \frac{26\theta(t_{23})}{2} - (0, 0, 1) S \frac{26\theta(t_{23})}{2} \right) \otimes \left(C \frac{23\theta(t_{22})}{2} + (0, 0, 1) S \frac{23\theta(t_{22})}{2} \right) \\
 &\quad \otimes \left(C \frac{25\theta(t_{21})}{2} - (0, 0, 1) S \frac{25\theta(t_{21})}{2} \right) \\
 &= \left(\left(C \frac{26\theta(t_{23})}{2} C \frac{23\theta(t_{22})}{2} + S \frac{26\theta(t_{23})}{2} S \frac{23\theta(t_{22})}{2} \right) \right. \\
 &\quad \left. + (0, 0, C \frac{26\theta(t_{23})}{2} S \frac{23\theta(t_{22})}{2} - S \frac{26\theta(t_{23})}{2} C \frac{23\theta(t_{22})}{2}) \right)
 \end{aligned}$$

$$\begin{aligned}
& \otimes (C \frac{25\theta(t_{21})}{2} - (0, 0, 1)S \frac{25\theta(t_{21})}{2}) \\
= & C \frac{23\theta(t_{22}) - 26\theta(t_{23})}{2} + (0, 0, S \frac{23\theta(t_{22}) - 26\theta(t_{23})}{2}) \\
& \otimes (C \frac{25\theta(t_{21})}{2} - (0, 0, S \frac{25\theta(t_{21})}{2})) \\
= & ((C \frac{23\theta(t_{22}) - 26\theta(t_{23})}{2} C \frac{25\theta(t_{21})}{2} + S \frac{23\theta(t_{22}) - 26\theta(t_{23})}{2} S \frac{25\theta(t_{21})}{2}) + \\
& (0, 0, -C \frac{23\theta(t_{22}) - 26\theta(t_{23})}{2} S \frac{25\theta(t_{21})}{2} + S \frac{23\theta(t_{22}) - 26\theta(t_{23})}{2} C \frac{25\theta(t_{21})}{2})) \\
= & C \frac{23\theta(t_{22}) - 26\theta(t_{23}) - 25\theta(t_{21})}{2} + (0, 0, S \frac{23\theta(t_{22}) - 26\theta(t_{23}) - 25\theta(t_{21})}{2}) \\
= & (C \frac{23\theta(t_{22}) - 26\theta(t_{23}) - 25\theta(t_{21})}{2} + (0, 0, 1)S \frac{23\theta(t_{22}) - 26\theta(t_{23}) - 25\theta(t_{21})}{2}),
\end{aligned}$$

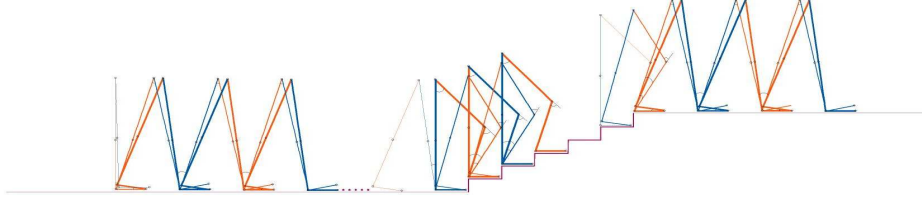


FIGURE 3

$$\begin{aligned}
Q_{W_3} &= Q_{W_{3p_3}} \otimes Q_{W_{3p_2}} \otimes Q_{W_{3p_1}} \\
&= (C \frac{\theta_{30}(t_{33})}{2} - (0, 0, 1)S \frac{\theta_{30}(t_{33})}{2}) \otimes (C \frac{\theta_{33}(t_{32})}{2} + (0, 0, 1)S \frac{\theta_{33}(t_{32})}{2}) \\
&\quad \otimes (C \frac{\theta_{31}(t_{31})}{2} - (0, 0, 1)S \frac{\theta_{31}(t_{31})}{2}) \\
&= (C \frac{\theta_{33}(t_{32}) - \theta_{30}(t_{33}) - \theta_{31}(t_{31})}{2} + (0, 0, 1)S \frac{\theta_{33}(t_{32}) - \theta_{30}(t_{33}) - \theta_{31}(t_{31})}{2}).
\end{aligned}$$

Quaternion form of WS is

$$\begin{aligned}
 Q_{WS} &= Q_{3pWS} \otimes Q_{2pWS} \otimes Q_{1pWS} \\
 &= \left(C \frac{(n+1)2\theta(t_{(n+1)2})}{2} - (0, 0, 1) S \frac{(n+1)2\theta(t_{(n+1)2})}{2} \right) \\
 &\quad \otimes \left(C \frac{(n+1)3\theta(t_{(n+1)2})}{2} + (0, 0, 1) S \frac{(n+1)3\theta(t_{(n+1)2})}{2} \right) \\
 &\quad \otimes \left(C \frac{(n+1)5\theta(t_{(n+1)1})}{2} - (0, 0, 1) S \frac{(n+1)5\theta(t_{(n+1)1})}{2} \right) \\
 &= \frac{\left(C \frac{(n+1)3\theta(t_{(n+1)2}) - (n+1)2\theta(t_{(n+1)2}) - (n+1)5\theta(t_{(n+1)1})}{2} \right. \\
 &\quad \left. + (0, 0, 1) S \frac{(n+1)3\theta(t_{(n+1)2}) - (n+1)2\theta(t_{(n+1)2}) - (n+1)5\theta(t_{(n+1)1})}{2} \right)}.
 \end{aligned}$$

Similarly, quaternion form of S_1 and ${}_2S$ are as below:

$$\begin{aligned}
 Q_{S_1} &= Q_{S_{1p_5}} \otimes Q_{S_{1p_4}} \otimes Q_{S_{1p_3}} \otimes Q_{S_{1p_2}} \otimes Q_{S_{1p_1}} \\
 &= \left(C \frac{\theta_{(n+2)4}(t_{(n+2)3})}{2} - (0, 0, 1) S \frac{\theta_{(n+2)4}(t_{(n+2)3})}{2} \right) \\
 &\quad \otimes \left(C \frac{\theta_{(n+2)3}(t_{(n+2)3})}{2} + (0, 0, 1) S \frac{\theta_{(n+2)3}(t_{(n+2)3})}{2} \right) \\
 &\quad \otimes \left(C \frac{\theta_{(n+2)2}(t_{(n+2)2})}{2} - (0, 0, 1) S \frac{\theta_{(n+2)2}(t_{(n+2)2})}{2} \right) \\
 &\quad \otimes \left(C \frac{\theta_{(n+2)1}(t_{(n+2)2})}{2} + (0, 0, 1) S \frac{\theta_{(n+2)1}(t_{(n+2)2})}{2} \right) \\
 &\quad \otimes \left(C \frac{\theta_{(n+2)1}(t_{(n+2)1})}{2} - (0, 0, 1) S \frac{\theta_{(n+2)1}(t_{(n+2)1})}{2} \right) \\
 &= \frac{\left(C \frac{\theta_{(n+2)3}(t_{(n+2)3}) + \theta_{(n+2)1}(t_{(n+2)2})}{2} \right. \\
 &\quad \left. \frac{\theta_{(n+2)4}(t_{(n+2)3}) + \theta_{(n+2)2}(t_{(n+2)2}) + \theta_{(n+2)1}(t_{(n+2)1})}{2} \right. \\
 &\quad \left. + (0, 0, 1) S \left(\frac{\theta_{(n+2)3}(t_{(n+2)3}) + \theta_{(n+2)1}(t_{(n+2)2})}{2} \right. \right. \\
 &\quad \left. \left. - \frac{\theta_{(n+2)4}(t_{(n+2)3}) + \theta_{(n+2)2}(t_{(n+2)2}) + \theta_{(n+2)1}(t_{(n+2)1})}{2} \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 Q_{{}_2S} &= Q_{5p{}_2S} \otimes Q_{4p{}_2S} \otimes Q_{3p{}_2S} \otimes Q_{2p{}_2S} \otimes Q_{1p{}_2S} \\
 &= \left(C \frac{(n+3)2\theta(t_{(n+3)3})}{2} - (0, 0, 1) S \frac{(n+3)2\theta(t_{(n+3)3})}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
& \otimes \left(C \frac{(n+3)3\theta(t_{(n+3)3})}{2} + (0, 0, 1) S \frac{(n+3)3\theta(t_{(n+3)3})}{2} \right) \\
& \otimes \left(C \frac{(n+3)4\theta(t_{(n+3)2})}{2} - (0, 0, 1) S \frac{(n+3)4\theta(t_{(n+3)2})}{2} \right) \\
& \otimes \left(C \frac{(n+3)5\theta(t_{(n+3)2})}{2} + (0, 0, 1) S \frac{(n+3)5\theta(t_{(n+3)2})}{2} \right) \\
& \otimes \left(C \frac{(n+3)5\theta(t_{(n+3)1})}{2} - (0, 0, 1) S \frac{(n+3)5\theta(t_{(n+3)1})}{2} \right) \\
= & \left(C \frac{(n+3)3\theta(t_{(n+3)3}) + (n+3)5\theta(t_{(n+3)2})}{2} \right. \\
& \left. - \frac{(n+3)2\theta(t_{(n+3)3}) + (n+3)4\theta(t_{(n+3)2}) + (n+3)5\theta(t_{(n+3)1})}{2} \right) \\
& + (0, 0, 1) S \frac{(n+3)3\theta(t_{(n+3)3}) + (n+3)5\theta(t_{(n+3)2})}{2} \\
& \left. - \frac{(n+3)2\theta(t_{(n+3)3}) + (n+3)4\theta(t_{(n+3)2}) + (n+3)5\theta(t_{(n+3)1})}{2} \right).
\end{aligned}$$

Finally, quaternion form of SW is:

$$\begin{aligned}
Q_{SW} &= Q_{SW_{p_5}} \otimes Q_{SW_{p_4}} \otimes Q_{SW_{p_3}} \otimes Q_{SW_{p_2}} \otimes Q_{SW_{p_1}} \\
&= \left(C \frac{\theta_{(n+m+1)0}(t_{(n+m+1)4})}{2} - (0, 0, 1) S \frac{\theta_{(n+m+1)0}(t_{(n+m+1)4})}{2} \right) \\
& \otimes \left(C \frac{\theta_{(n+m+1)3}(t_{(n+m+1)3})}{2} + (0, 0, 1) S \frac{\theta_{(n+m+1)3}(t_{(n+m+1)3})}{2} \right) \\
& \otimes \left(C \frac{\theta_{(n+m+1)2}(t_{(n+m+1)2})}{2} - (0, 0, 1) S \frac{\theta_{(n+m+1)2}(t_{(n+m+1)2})}{2} \right) \\
& \otimes \left(C \frac{\theta_{(n+m+1)1}(t_{(n+m+1)2})}{2} + (0, 0, 1) S \frac{\theta_{(n+m+1)1}(t_{(n+m+1)2})}{2} \right) \\
& \otimes \left(C \frac{\theta_{(n+m+1)1}(t_{(n+m+1)1})}{2} - (0, 0, 1) S \frac{\theta_{(n+m+1)1}(t_{(n+m+1)1})}{2} \right) \\
= & \left(C \frac{\theta_{(n+m+1)3}(t_{(n+m+1)3}) + \theta_{(n+m+1)1}(t_{(n+m+1)2})}{2} \right. \\
& \left. - \frac{\theta_{(n+m+1)0}(t_{(n+m+1)4}) + \theta_{(n+m+1)2}(t_{(n+m+1)2}) + \theta_{(n+m+1)1}(t_{(n+m+1)1})}{2} \right) \\
& + (0, 0, 1) S \frac{\theta_{(n+m+1)3}(t_{(n+m+1)3}) + \theta_{(n+m+1)1}(t_{(n+m+1)2})}{2} \\
& \left. - \frac{\theta_{(n+m+1)0}(t_{(n+m+1)4}) + \theta_{(n+m+1)2}(t_{(n+m+1)2}) + \theta_{(n+m+1)1}(t_{(n+m+1)1})}{2} \right).
\end{aligned}$$

4. Conclusion

In this study, matrices of the walking and climbing the stairs motion of the model, which is defined as $2 \times 3R$ robotics modelling, have been obtained. The chain of motions belonging to the model is important because the fixed frame's placement could be re-fixed in all stages. The motion is independent of the order of the indic in many stages. Because of this, the order relation, which is well-matched with the given indic, was defined on the time line. The matrices of the motion of W_1 , ${}_2W$, W_3 , WS , S_1 , ${}_2S$ and SW were obtained. Consequently, we wrote quaternion forms of W_1 , ${}_2W$, W_3 , WS , S_1 , ${}_2S$ and SW .

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