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SELECTED PROBLEMS OF THE SECOND NATIONAL MATHEMATICS OLYMPIAD

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Introduction

To encouraging excellence in the educational environment in Pakistan, the Abdus Salam School of Mathematical Sciences (ASSMS) (formerly known as School of Mathematical Sciences (SMS)) has taken the initiative to provide a world class coaching to the young Pakistani students participating in the different mathematics contests at National and International level.

National Mathematics Olympiad (NMO) was first held in August 2004 and is conducted on annual basis. A large number of young students take part in this competition every year since. It is organized in several rounds.

This article has two parts. The first part consists of all the selected problems which appeared in different rounds of the Second National Mathematics Olympiad together with the problems given in the qualifying tests for the Pakistani National Team participating in the International Mathematical Olympiad (IMO).

This collection is intended as practice for the serious student who wishes to improve his/her performance in National and International Contests. We do hope that mathematics teachers will also be involved to come up with new original problems.

The second part consists of the solutions of the problems which appeared in different rounds of the First and Second National Mathematics Olympiad together with solutions of the problems given in the qualifying tests for the

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Pakistani National Team participating in IMO.

The Abdus Salam School of Mathematical Sciences invites solutions and also new original problems from mathematics teachers and students. Please send your solutions and problems to: saifullahkhalid75@yahoo.com. Best solutions with the names of the solvers will be published in the next issue of the MATH TRACK and will also be posted on the website: www.sms.edu.pk.

I. Selected Problems of the Second National Mathematics Olympiad

In this part, we give selected problems of tests taken for Second National Mathematics Olympiad together with the problems given in the qualifying tests for the Pakistani National Team participating in IMO.

Tests Taken for Second National Mathematics Olympiad at Different Cities of Pakistan

ISLAMABAD/QUETTA - 30.09.2006

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. The rectangle $ABCD$ has $|AB| = |CD| = 3|AD| = 3|BC|$. On the side AB we take the point E such that $|AE| = 2|EB|$. Lines AC and DE meet at F. Find the measure of $\angle AFD$.

Problem.2. Consider the equation $\frac{xy-1}{x+y} = k$, where k is a fixed positive integer. Show that, for any given k, this equation has a finite, not null number of solutions with x, y positive integers.

Problem.3. Show that, for any positive integer n, there exist positive integer values k such that $k^2 + 1$ has more than n distinct divisors.

Problem.4. The price of an item dropped on September 1^{st} (as compared to its August value) by $x\%$, but subsequently increased on October 1^{st} by $y\%$ (as compared to its September value). It turns out that on October 1^{st} the price in fact dropped by $(y - x)\%$ (as compared to its August value). Find x and y knowing that they are both positive integers.

Problem.5. Around a circle are written 1,000 real numbers, of total sum 1,000. Show that we can find among them a sequence of three consecutively contiguous numbers, with sum not more than 3.

LAHORE - 01.10.2006

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. Triangle ABC is isosceles with $|CA| = |CB|$ and $\angle C = 120^0$. Points D and E are taken on the segment AB such that $|AD| = |DE| = |EB|$. Find the measures of the angles of triangle CDE.

Problem.2. Let m, n be relatively prime positive integers (this meaning that $\gcd(m, n) = 1$). Show that

$$m^3 + mn + n^3 \quad \text{and} \quad mn(m + n)$$

are also relatively prime.

Problem.3. Find the minimum value of the expression

$$t^2(xy + yz + zx) + 2t(x + y + z)$$

if x, y, z, t are real numbers of absolute values not greater than 1.

Problem.4. Given that $0 < a < 1$, $0 < b < 1$ and $0 < c < 1$, show that at least one of the numbers $a(1 - b)$, $b(1 - c)$, $c(1 - a)$ does not exceed $\frac{1}{4}$.

Problem.5. Let us consider $4n$ points on a line, such that $2n$ of them are colored in white and (the other) $2n$ are colored in black (this being any completely random colorization). Prove that there exists among them a sequence of $2n$ consecutive points, such that n of them are white and n black.

KARACHI - 05.10.2006

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. In the cyclic quadrilateral $ABCD$, the diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E. Show that $|CE| = |CA|$, if and only if $|DE| = |AB|$.

Problem.2. Find all pairs of positive integers a, b such that

$$ab = \gcd(a, b) + \text{lcm}(a, b).$$

Problem.3. For real numbers $a \geq b \geq c$, prove the inequality

$$a^2b + b^2c + c^2a \geq ab^2 + bc^2 + ca^2.$$

Problem.4. Given the real numbers a, b, show that among the equations

$$x^2 + 2ax + b = 0, \quad ax^2 + 2bx + 1 = 0, \quad bx^2 + 2x + a = 0,$$

at least one has real roots.

Problem.5. The entries of a 9×9 array are the positive integers from 1 to 81, each taken once. Prove that it is impossible for each row to have the product of the numbers in it equal to the product of the numbers in the column of same rank with the row.

(It can be found, with the aid of a computer, that in the case of an 8×8 array filled with the positive integers from 1 to 64, it may happen that each row has the product of its elements equal to the product of the elements in the column of same rank with the row).

SAKARDU - 07.10.2006

Time allowed: $3\frac{1}{2}$ Hours

Problem.1. Let ABC be an acute-angled triangle, M the midpoint of BC, and D a point on the segment AM, such that $\angle BDC + \angle BAC = 180^0$. Show that $|AB| \cdot |CD| = |AC| \cdot |BD|$.

Problem.2. Prove that, in any convex quadrilateral, (at least) one diagonal is longer than one-fourth of the perimeter.

Problem.3. Let n be a positive integer. Find (with proof) all positive integer solutions for the equation

$$7 \cdot 4^n = a^2 + b^2 + c^2 + d^2.$$

Problem.4. If p, q are prime numbers, and $\frac{p}{x} + \frac{q}{y}$ is an integer, where $x < p$, $y < q$ are positive integers, then show it follows with necessity that $x = y$.

Problem.5. Prove that, for any positive integer $n \geq 5$, the set $X_n = \{1, 2, \ldots, n\}$ can be partitioned into two subsets S_n and P_n, such that the sum of the elements of S_n be equal to the product of the elements of P_n.

Test for Second National Mathematics Olympiad

Problem.1. A function f is defined for all positive integers and it satisfies $f(1) = 2006$ and $f(1) + f(2) + \ldots\ldots + f(n) = n^2 f(n)$ for all $n > 1$. Determine $f(2005)$?

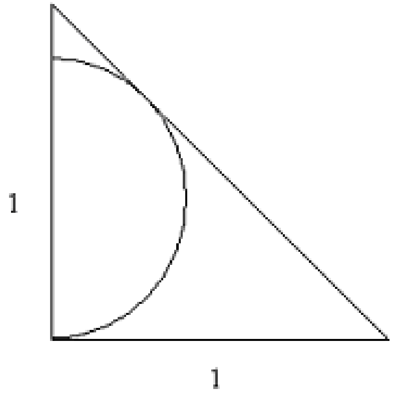
Problem.2. In how many different ways can seven numbers be chosen from the numbers 1 to 9, inclusive, so that the seven numbers have a sum which is a multiple of 3 ?

Problem.3. Find all pairs of positive integers (a, b) such that

$$ab = \text{gcd}(a, b) + \text{lcm}(a, b)$$

Problem.4. Triangle ABC is isosceles with $AC = BC$ and $\angle C = 120^0$. Points D and E are chosen on segment AB so that $|AD| = |DE| = |EB|$. Find the degree measure of all the angles of triangle CDE.

Problem.5. A semi circle is inscribed in an isosceles right-angled triangle as shown.



Find the radius of the semi circle?

IMO Training Camp Tests

First Test:

Problem.1. Let ABC be a triangle. Show that $\angle BAC = 90^0$ if and only if

$$\cos B + \cos C = \frac{b + c}{a}$$

Problem.2. Let a, b, c be positive numbers such that

$$\frac{a(b-c)}{b+c}+\frac{b(c-a)}{c+a}+\frac{c(a-b)}{a+b}=0$$

show that $(a-b)(b-c)(c-a)=0$.

Problem.3. In a group of 3 persons, the average of the age of any 2 persons is the age of the third person. Show that the total age of the three persons is divisible by 3.

Second Test:

Problem.1. Let a, b, c be positive integers such that

$$\frac{a+b}{bc}=\frac{b+c}{ca}=\frac{c+a}{ab}$$

show that $a=b=c$.

Problem.2. We are given an isosceles triangle ABC such that $AB = AC$ and $\angle BAC = 20^0$. Let M be the foot of the altitude from C and N be a point on the side AC such that $CN = \frac{1}{2}BC$. Find the size of the angle $\angle AMN$.

Problem.3. Find the least value of the sum

$$|x-1|+|x-2|+........+|x-10|$$

when x runs over the set of all real numbers.

First Selection Test for IMO (January 25, 2007)

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. Let ABC be a triangle and D be the point where the angle bisector of $\angle A$ meets BC. Prove that the line BC, the perpendicular bisector of AD and the perpendicular at A on AC are concurrent if and only if $\angle B$ is a right angle.

Problem.2. Prove the following inequality

$$(ab+bc+ca)(1+18abc) \geq 15abc$$

for a, b, c non-negative real numbers with $a+b+c=1$.
State, with proof, the case(s) of equality.

Problem.3. Solve in non-negative integer numbers

$$x^2 + y^4 + z^6 = 2^{1111}$$

Problem.4. In a cricket league, n teams play a complete round-robin tournament, with 2 points awarded for a win, 1 point for a tie and 0 points for a loss. Determine the admissible n such that the tournament could end with each team having won exactly half of its points against the last 6 ranking teams (this being true for these teams also, each of them having won exactly half of its points against the other 5).

Second Selection Test for IMO (January 26, 2007)

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. Triangle ABC is isosceles with $CA = CB$, I is its incenter and M is the meeting point of CI and AB.
1. Show that the circumcircle Γ of triangle AIB is tangent to AC, BC;
2. Show that for any point P on the arc AIB of Γ the following holds

$$\frac{PC}{PM} = \frac{AC + BC}{AB};$$

3. Do the results from points 1. and/or 2. remain true for a scalene triangle ABC?

Problem.2. Let $P \in \mathbb{Z}[X]$ be a polynomial of degree $d \geq 2$. Prove that the sequence $(a_n)_{n\geq 1}$ given by $a_n = P(n)$ may contain d (but cannot contain more than d) successive terms in arithmetical progression.

Problem.3. Given integer numbers x_s, $1 \leq s \leq n$, of distinct remainders when divided by n, determine the remainder of division by $n \geq 3$ for the expression

$$E = \sum_{1\leq i<j<k\leq n} x_i x_j x_k$$

Problem.4. Determine the values n for which a regular polygon with n sides (n-gon) can be triangulated by non-crossing diagonals in such a way that all triangles in the triangulation be isosceles.

Third Selection Test for IMO (April 27, 2007)

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. Show that, for $n \geq 4$, the equation

$$1 = \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}$$

has atleast two sets of integral solutions, satisfying $1 < x_1 < x_2 < ...... < x_n$.

Problem.2. Let l_1, l_2 be two lines in the plane and M a variable point on the line l_1. One considers the circle Γ of center M which is tangent to l_2 and the circle Γ_1 tangent to l_1, l_2 and Γ. Let T be the touching point of Γ and Γ_1 . Find the locus of T when M varies on l_1.

Problem.3. There are 121 people at a party. Show that either there are 13 people who mutually do not know each other, or there is a person who knows at least 10 others. Find and prove a general statement.

Fourth Selection Test for IMO (April 28, 2007)

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. A sequence of real numbers $1, a_0, a_1, a_2, ...$ is defined by the formula

$$a_{n+1} = \lfloor a_n \rfloor \cdot \langle a_n \rangle$$

for $n \geq 0$; here a_0 is an arbitrary real number, $\lfloor a_n \rfloor$ denotes the greatest integer not exceeding a_n, and $\langle a_n \rangle = a_n - \lfloor a_n \rfloor$. Prove that $a_n = a_{n+2}$ for n sufficiently large.

Problem.2. A convex polygon $\mathcal{P}$ is *decomposable* if there is a point O and two other convex polygons $\mathcal{Q}$ and $\mathcal{R}$ such that any point A in $\mathcal{P}$ can be written as

$$\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}$$

where B is in $\mathcal{Q}$ and C is in $\mathcal{R}$ and no side of $\mathcal{Q}$ is parallel to a side of $\mathcal{R}$. Find all regular polygons which are decomposable.

Problem.3. Let $P(X) \in \mathbb{Z}[X]$ a polynomial of degree $n \geq 4$ and consider the image set $A = \{P(k) \setminus k \in \mathbb{Z}\}$. If $\{0, 1, 2, ..., n-1\}$ is a subset of A, prove that $\{n, n+1, ..., 2n-1\}$ is disjoint from A.

II. Solutions of the Selected Problems of First and Second National Mathematics Olympiad

In this part we give solutions of the selected problems of tests taken for First and Second National Mathematics Olympiad together with solutions of the problems given in the qualifying tests for the Pakistani National Team participating in IMO.

Solutions of the Selected Problems of First National Mathematics Olympiad.

Following are the solutions of the selected problems of tests taken for First National Mathematics Olympiad at different cities of Pakistan.

Islamabad Center

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. Let x be a real number such that $x^4+\frac{1}{x^4}=2$. Find the possible values of $x+\frac{1}{x}$.

Solution. Let $x+\frac{1}{x}=t$. We have $x^2+\frac{1}{x^2}=(x+\frac{1}{x})^2-2=t^2-2$ and $x^4+\frac{1}{x^4}=(x^2+\frac{1}{x^2})^2-2=t^4-4t^2+2$. It follows that $t^4-4t^2=0$, and hence $t=\pm 2$ (since $t=0$ leads to $x^2+1=0$, a contradiction).

Problem.2. How many 4-digit numbers are there which begin with 1 and have exactly two identical digits? (Examples are $1557, 1030, 1321$.)

Solution. If the two identical digits are 1's, we have three cases: $\overline{11ab}$, $\overline{1a1b}$, $\overline{1ab1}$. In each case, a can be chosen in 9 ways ($a\neq 1$) and b in 8 ways ($b\neq 1,\ b\neq a$), so we have $3\cdot 9\cdot 8=216$ numbers.
If the two identical digits are not 1's, we again have three cases: $1aab$, $1aba$, $1baa$. Similarly, we obtain 216 numbers, hence the answer is 432.

Problem.3. Let P be a point in the interior of the triangle ABC. The reflections of P across the midpoints of the sides BC, CA, AB, are P_A, P_B, and P_C respectively. Prove that the lines AP_A, BP_B and CP_C are concurrent.

Solution. Observe that the line segments BC and PP_A have the same midpoint, hence $BPCP_A$ is a parallelogram. Similarly, $BPAP_C$ is a parallelogram and it follows that ACP_AP_C is parallelogram as well. Therefore, the midpoint of AP_A coincides with the midpoint of CP_C. In the same way, the midpoint of AP_A coincides with the midpoint of BP_P, hence the three lines are concurrent.

Karachi Center

Time allowed: $3\frac{1}{2}$ Hours

Problem.1. Find all positive integers n such that the number n^8+n^4+1 is a prime.

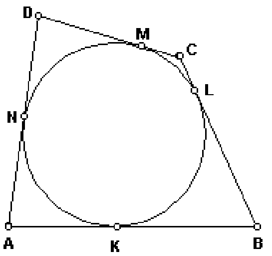
Solution. Observe that $n^8+n^4+1=(n^4+1)^2-n^4=(n^4-n^2+1)(n^4+n^2+1)$. If n^8+n^4+1 were a prime, then $n^4-n^2+1=1$, hence $n=1$.

Problem.2. Let $n \geq 2$ be an integer and let $x_1, x_2, \ldots\ldots., x_n \in \{-1, 1\}$ such that $x_1x_2+x_2x_3+\ldots\ldots.+x_{n-1}x_n+x_nx_1=0$. Prove that n is divisible by 4.

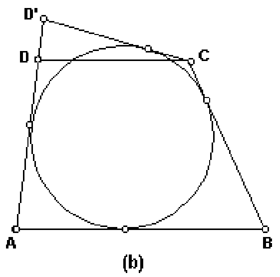
Solution. Each of the products $x_1x_2, x_2x_3, \ldots\ldots, x_nx_1$ equals ± 1, so if they add up to 0, n must be even. Moreover, exactly $\frac{n}{2}$ products are equal to -1, therefore multiplying them out, we obtain $(-1)^{\frac{n}{2}}$. But $(x_1x_2)(x_2x_3)\ldots\ldots(x_nx_1)=(x_1x_2\ldots..x_n)^2=1$, hence $\frac{n}{2}$ is even, that is, n is divisible by 4.

Problem.3. Let $ABCD$ be a convex quadrilateral. Prove that there exists a circle in its interior touching all its sides if and only if $AB+CD=AD+BC$.

Solution.



(a)



(b)

Suppose there is a circle touching the sides AB, BC, CD, DA at K, L, M, N, respectively (see Figure (a)). Then we have $AK=AN$, $BL=BK$, etc. Adding up yields the result. Conversely, assume that

$$AB+CD=AD+BC \qquad (1).$$

We can always draw a circle tangent to three of the sides of $ABCD$, say AB, BC, and AD (the center of this circle is the intersection between the angle bisectors of $\angle A$ and $\angle B$). Suppose, by way of contradiction that this circle is not tangent to CD and take the point D' on AD such that CD' is tangent to the circle (see Figure (b)). From the first part, we deduce that $AB + CD' = AD' + BC$. Subtracting (1) from this yields $DD' = CD + CD'$, a contradiction.

Lahore Center

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. Let m, n be positive integers such that $\frac{7}{10} < \frac{m}{n} < \frac{11}{15}$. Find the smallest possible value of n.

Solution. We will show that the smallest possible value of n is 7. Indeed, suppose $n \leq 6$. Then n is a divisor of 60 and we can write $\frac{m}{n} = \frac{k}{60}$, where k is a composite number. But $\frac{7}{10} = \frac{42}{60}$ and $\frac{11}{15} = \frac{44}{60}$, implying $k = 43$, a contradiction. For $n = 7$, we find, by inspection, $m = 5$.

Problem.2. Denote by P the perimeter of triangle ABC. If M is a point in the interior of the triangle, prove that

$$\frac{1}{2}P < MA + MB + MC < P.$$

Solution. We have by triangle's inequality $MA + MB > AB$, etc. Adding up yields the first inequality.
For the second one, we first prove that $MA+MB < CA+CB$. Suppose AM intersects the side BC at N. We have $CA+CB = CA+CN+NB > AN+NB$. Now, $AN + NB = AM + MN + NB > AM + MB$, as desired. Similarly, we have $MA + MC < BA + BC$ and $MB + MC < AB + AC$. Adding up, we obtain $MA + MB + MC < P$.

Problem.3. Prove that if an arithmetical progression of positive integers contains a square, then it contains infinitely many squares.

Solution. Let r be the common difference of the progression and let a^2 be one of its terms. Then the following terms are $a^2 + r$, $a^2 + 2r$, $a^2 + 3r$, etc. Eventually, we reach the term $a^2 + (2a + r)\ r = (a + r)^2$, which is again a square. Repeating the procedure, we see that all numbers $(a+2r)^2$, $(a+3r)^2$, etc are among the progression's terms.

Peshawar Center

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. Let a, b, c be real numbers. Prove the inequalities:
(a) $a^2 + b^2 + c^2 \geq ab + bc + ca$; (b) $a^4 + b^4 + c^4 \geq abc(a + b + c)$.

Solution. (a) Since $(a - b)^2 \geq 0$, we have $a^2 + b^2 \geq 2ab$. Similarly, $b^2 + c^2 \geq 2bc$, $a^2 + c^2 \geq 2ac$. Adding up these inequalities yields the result.

(b) Using the inequality from (a), we have $a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$. But $a^2b^2 + b^2c^2 + c^2a^2 = (ab)^2 + (bc)^2 + (ca)^2$ and we apply (a) again:

$$(ab)^2 + (bc)^2 + (ca)^2 \geq (ab)(bc) + (bc)(ca) + (ca)(ab) = abc(a + b + c).$$

Problem.2. Prove that the product of four consecutive positive integers:
(a) is divisible by 24 ; (b) is never a perfect square.

Solution. (a) Two out of four consecutive integers are even. Moreover, one of them is divisible by 4, hence the product is divisible by 8. One out of three consecutive integers is divisible by 3, hence the product is divisible by 24.

(b) Let $n, n + 1, n + 2, n + 3$ be the positive integers. Observe that

$$n(n+1)(n+2)(n+3) = [n(n+3)][(n+1)(n+2)] = (n^2+3n)(n^2+3n+2) = k(k+2),$$

where $k = n^2 + 3n$. Butt $k(k + 2) = k^2 + 2k$ is never a square, since $k^2 < k^2 + 2k < (k + 1)^2$.

Problem.3. The side length of the equilateral triangle ABC equals l. The point P lies in the interior of ABC and the distances from P to the triangle's sides are $1, 2, 3$. Find the possible values of l.

Solution. The area of ABC equals $S_{ABC} = \frac{l^2\sqrt{3}}{4}$. On the other hand, we have $S_{ABC} = S_{ABP} + S_{BCP} + S_{CAP} = \frac{l}{2}(1 + 2 + 3) = 3l$. We obtain $l = 4\sqrt{3}$.

Sakardu Center

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. How many solutions has the equation $A \cup B = \{1, 2, 3, 4\}$?
(two solutions (A_1, B_1) and (A_2, B_2) are different iff $A_1 \neq A_2$ or $B_1 \neq B_2$;

for instance, $A_1 = \{1,3,4\}$, $B_1 = \{1,2,4\}$ and $A_2 = \{1,2,4\}$, $B_2 = \{1,3,4\}$ are different solutions).

Solution. A solution is defined by an ordered partition $X \coprod Y \coprod Z = (A\backslash B) \coprod (A \cap B) \coprod (B\backslash A)$ of the given set $S = \{1,2,3,4\}$. And these ordered partitions correspond to bijections $f : S \to \{1,2,3\}$. For instance, the solution $A_1 = \{1,3,4\}$, $B_1 = \{1,2,4\}$ is defined by the ordered partition $\{3\} \coprod \{1,4\} \coprod \{2\}$, which correspond to the function $f(3) = 1,\ f(1) = f(4) = 2,\ f(2) = 3$. Hence the number of solutions equals the number of functions $= 3^4 = 81$.

Problem.2. Can you find a polynomial $f \in Z[X]$ such that $f(Z) \subset N$ and f takes the value -2005?

Solution. There are many examples ; for instance take $f(0) = 0 = f(1)$ and a large negative value for $f(\frac{1}{2})$:

$$f(x) = -8020x^2 + 8020x.$$

Problem.3. Compute $\cos \frac{\pi}{5}$.

First Solution. Consider the regular pentagon $ABCDE$ and consider $F = AC \cap BD$. Computing angles, we get $AB \equiv AF$ and $\triangle ABC \sim \triangle BFC$, hence

$$\frac{(a+x)}{a} = \frac{a}{x}$$

where $a = AB,\ x = BF$.
Since $\angle ABF = \angle AFB = \angle FBC + \angle FCB = 2\angle DBC = \frac{\pi}{5}$, we obtain $\cos \frac{\pi}{5} = \frac{x}{2a} = \frac{(-1+\sqrt{5})}{2}$.

Second Solution. $(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5})^5 = -1$, hence, for $x = \cos \frac{\pi}{5}$, the imaginary part of the equation becomes:

$$16x^4 - 12x^2 + 1 = 0 \quad \text{or} \quad x = \pm i \frac{(-1+\sqrt{5})}{2}.$$

Solutions of the Selected Problems of Second National Mathematics Olympiad.

Following are the solutions of the selected problems of tests taken for Second National Mathematics Olympiad at different cities of Pakistan.

ISLAMABAD/QUETTA - 30.09.2006.

Problem.1. The rectangle $ABCD$ has $|AB| = |CD| = 3|AD| = 3|BC|$. On the side AB we take the point E such that $|AE| = 2|EB|$. Lines AC and DE meet at F. Find the measure of $\angle AFD$.

First Solution. Consider the points X, Z, symmetrical to A, B with respect to the line DC, and the point Y on XZ such that $|YZ| = 2|XY|$. Clearly, $|AY| = |YC| = |DE|$, $\angle AYC = 90^0$ and YC parallel to DE, hence $\angle AFD = \angle ACY$. But triangle AYC is isosceles and right-angled in Y, therefore $\angle ACY = 45^0$ and so is $\angle AFD = 45^0$.

Second Solution. Trigonometrical solution: $\tan(\angle ADE) = 2$, $\tan(\angle DAC) = 3$, therefore using the formula

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

we get $\tan(\angle AFD) = \frac{2+3}{2\cdot 3-1} = 1$, therefore $\angle AFD = 45^0$.

Problem.2. Consider the equation

$$\frac{xy-1}{x+y} = k,$$

where k is a fixed positive integer. Show that, for any given k, this equation has a finite, not null number of solutions with x, y positive integers.

Solution. The equation can be written $(x-k)(y-k) = k^2+1$. In order to have solution(s) we need $x-k$ and $y-k$ of same sign. Now, if both $x \le k$ and $y \le k$, then $(x-k)(y-k) = (k-x)(k-y) < k \cdot k = k^2 < k^2+1$, therefore we need to have $x > k$ and $y > k$. But if either of them is greater than k^2+k+1, again there is no solution, so the number of possible solutions is finite. As k^2+k+1 and $k+1$ are a solution, we have answered affirmatively to the stated assertion of the problem.

Problem.3. Show that, for any positive integer n, there exist positive integer values k such that k^2+1 has more than n distinct divisors.

Solution. For any k, k^2+1 is never divisible by 3. We shall use the observation that $m^3+1 = (m+1)[(m+1)(m-2)+3]$, so a common divisor of the two factors can be at most 3, but this cannot happen when $m = k^2$, therefore the two factors are relatively prime. Moreover, for $k > 1$, both factors are greater than 1.
Let us now take $k_1 = k^3$, then $k_1^2 + 1 = k^6 + 1 = (k^2+1)(k^4-k^2+1)$,

where the two factors have been seen to be relatively prime, so k_1^2+1 has at least one more prime divisor than k^2+1. Inductively we can build numbers $k_n = k_{n-1}^3 = k^{3^n}$, with k_n^2+1 having more than n distinct prime divisors when starting with $k > 1$.

Problem.4. The price of an item dropped on September 1st (as compared to its August value) by $x\%$, but subsequently increased on October 1st by $y\%$ (as compared to its September value). It turns out that on October 1st the price in fact dropped by $(y-x)\%$ (as compared to its August value). Find x and y knowing that they are both positive integers.

Solution. If the price of the item in August was p, then on September 1st it was $p - px/100 = p(1 - x/100)$, and on October 1st it was $p(1 - x/100) + p(1 - x/100)y/100 = p(1 - x/100)(1 + y/100)$. We are told that in fact this was $p - p(y - x)/100 = p(1 - y/100 + x/100)$, therefore $(100 - x)(100 + y) = 100(100 - y + x)$, or $xy - 200y + 200x = 0$, i.e. $(200 - x)(200 + y) = 40000$. Now, $40000 = 2^6 \cdot 5^4$, and $200 - x$ may only take integer values between 100 and 200, while $200 + y$ may only take integer values between 200 and 300, so the only possibility is $x = 40$ and $y = 50$.

Problem.5. Around a circle are written 1,000 real numbers, of total sum 1,000. Show that we can find among them a sequence of three consecutively contiguous numbers, with sum not more than 3.

Solution. Consider all sequences of three consecutively contiguous numbers; there are 1,000 such sequences, and each number occurs in exactly three of them, so the 1,000 partial sums together total up to $3 \cdot 1,000 = 3,000$. It will then exist (at least) one of the partial sums not greater than 3 (otherwise the sum of them all will be greater than 3,000).

LAHORE - 01.10.2006.

Problem.1. Triangle ABC is isosceles with $|CA| = |CB|$ and $\angle C = 120^0$. Points D and E are taken on the segment AB such that $|AD| = |DE| = |EB|$. Find the measures of the angles of triangle CDE.

Solution. There is a funny solution. Build the (unique) equilateral triangle $CD'E'$ with D', E' on AB. We have $\angle CAB = \angle CBA = 30^0$ and also $\angle ACD' = \angle BCE' = 30^0$, from the symmetry of the figure. But then triangles $AD'C$ and $BE'C$ are isosceles and so $|AD'| = |D'C| = |D'E'| = |E'C| = |E'B|$, therefore D', E' divide AB in thirds, hence are correspondingly identical to D, E. The answer is therefore that all angles of triangle CDE have measure 60^0.

Problem.2. Let m, n be relatively prime positive integers (this meaning that $\gcd(m, n) = 1$). Show that m^3+mn+n^3 and $mn(m+n)$ are also relatively prime.

Solution. It is immediate that mn and $m + n$ are relatively prime. Now, $m^3 + mn + n^3 = (m + n)^3 - 3mn(m + n) + mn$ will be relatively prime to $mn(m + n)$, as a prime p dividing $mn(m + n)$ either divides mn, and then cannot divide $(m + n)^3$, or divides $m + n$, and then cannot divide mn.

Problem.3. Find the minimum value of the expression $t^2(xy + yz + zx) + 2t(x + y + z)$, if x, y, z, t are real numbers of absolute values not greater than 1.

Solution. All we need to do is notice that our expression is equal to $(tx + 1)(ty + 1) + (ty + 1)(tz + 1) + (tz + 1)(tx + 1) - 3$. As the absolute values of tx, ty and tz will also be not greater than 1, it follows that the value of our expression will be greater than or equal to -3, with equality reached e.g. when $x = y = z = 1$, $t = -1$.

Problem.4. Given that $0 < a < 1$, $0 < b < 1$ and $0 < c < 1$, show that at least one of the numbers $a(1 - b)$, $b(1 - c)$, $c(1 - a)$ does not exceed $\frac{1}{4}$.

Solution. Let us estimate the product of the three expressions

$$[a(1 - b)][b(1 - c)][c(1 - a)] = [a(1 - a)][b(1 - b)][c(1 - c)] \leq \frac{1}{2^6} = \frac{1}{64}$$

as in general for $0 < x < 1$ one has $x(1 - x) \leq (\frac{1}{2})^2 = \frac{1}{4}$, from AM-GM inequality. Now it is clear that all three factors cannot exceed $\frac{1}{4}$, as then their product would exceed $\frac{1}{64}$.

Problem.5. Let us consider $4n$ points on a line, such that $2n$ of them are colored in white and (the other) $2n$ are colored in black (this being any completely random colorization). Prove that there exists among them a sequence of $2n$ consecutive points, such that n of them are white and n black.

Solution. Look at the sequences of the first $2n$ points (from the left), and the last $2n$ points. Clearly the distribution of white and black points is reversed within these two sequences; if we consider the difference δ between the number of white points and black points within a sequence of $2n$ consecutive points, then δ takes values of opposite signs for the two extreme sequences discussed above (if it is zero, then we are finished). Moreover, δ is always an even number. Now, if we "slide" the leftmost sequence by one point in turn to the right, the value of δ either remains the same, or changes by +2 or -2, according to the colors of the point that "leaves" the sequence and the point that "enters" the sequence. As we saw that the value of δ will change signs by the time we have reached the rightmost sequence, it follows that at some time in-between it passed through value zero. This is a striking example of "continuity in discrete argument".

KARACHI - 05.10.2006.

Problem.1. In the cyclic quadrilateral $ABCD$, the diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E. Show that $|CE| = |CA|$, if and only if $|DE| = |AB|$.

Solution. We have $|BC| = |CD|$ as chords subtended by equal arcs, as $\angle BAC = \angle DAC$, and also $\angle CDE = \angle CBA$ as both supplementary to $\angle CDA$. We will consider triangles ABC and EDC.

1. If $|DE| = |AB|$, then the triangles are congruent (side-angle-side), whence $|CE| = |CA|$.

2. If $|CE| = |CA|$, then triangle ACE is isosceles, and so $\angle DEC = \angle DAC = \angle BAC$, and again the triangles above are congruent (similar, with one corresponding side of equal measure), whence $|DE| = |AB|$.

In fact, things may be looked at from the point of view of a rotation of center C which brings B onto D.

Problem.2. Find all pairs of positive integers a, b such that

$$ab = \gcd(a, b) + \operatorname{lcm}(a, b).$$

Solution. All we need know is that $ab = \gcd(a, b) \times \operatorname{lcm}(a, b)$, which is readily proven: if we denote $d = \gcd(a, b)$ then taking $a' = a/d$, $b' = b/d$ we get a', b' relatively prime, so $l = \operatorname{lcm}(a, b) = da'b'$ and $(da')(db') = (d)(da'b')$.
Now that gives us that $d + l = dl$, or $(d-1)(l-1) = 1$, yielding $d = l = 2$ and the unique solution $a = b = 2$.

Problem.3. For real numbers $a \geq b \geq c$, prove the inequality

$$a^2b + b^2c + c^2a \geq ab^2 + bc^2 + ca^2.$$

First Solution. The imperial way to solving it is to realize that

$$(a^2b + b^2c + c^2a) - (ab^2 + bc^2 + ca^2) = (a - b)(b - c)(a - c) \geq 0.$$

Second Solution. Alternatively, if we denote $a = b + x$ and $c = b - y$, with x, y non-negative real numbers, we only need to replace the expressions for a, c in the inequality to trivially arrive at $xy^2 \geq -x^2y$, clearly true.

Problem.4. Given the real numbers a, b, show that among the equations

$$x^2 + 2ax + b = 0, \quad ax^2 + 2bx + 1 = 0, \quad bx^2 + 2x + a = 0,$$

at least one has real roots.

Solution. If $a = 0$ or $b = 0$ the conclusion is trivial, so assume $ab \neq 0$. Let us add up the discriminants of the three equations

$$(a^2 - b) + (b^2 - a) + (1 - ab) = \frac{1}{2}[(a - b)^2 + (a - 1)^2 + (b - 1)^2] \geq 0$$

therefore at least one needs be non-negative, and so its equation have real roots.

Problem.5. The entries of a 9×9 array are the positive integers from 1 to 81, each taken once. Prove that it is impossible for each row to have the product of the numbers in it equal to the product of the numbers in the column of same rank with the row.
(It can be found, with the aid of a computer, that in the case of an 8×8 array filled with the positive integers from 1 to 64, it may happen that each row has the product of its elements equal to the product of the elements in the column of same rank with the row).

Solution. Any "large" prime number p such that $1 < p \leq 81 < 2p$ will clearly appear only once in the array, and therefore will need to be situated on its main diagonal, otherwise its row will have the product of the numbers in it different than the product of the numbers in the column of same rank. But these prime numbers are 41, 43, 47, 53, 59, 61, 67, 71, 73 and 79, ten in all, and they cannot fit together in the nine positions available on the main diagonal of the array.

SAKARDU - 07.10.2006.

Problem.1. Let ABC be an acute-angled triangle, M the midpoint of BC, and D a point on the segment AM, such that $\angle BDC + \angle BAC = 180^0$. Show that $|AB| \cdot |CD| = |AC| \cdot |BD|$.

Solution. Extend AM beyond M to a point E such that $|MD| = |ME|$. Then $BDCE$ is a parallelogram, as its diagonals cut each other in half, so $\angle BDC = \angle BEC$. Therefore $\angle BEC + \angle BAC = 180^0$, hence $ABEC$ is a cyclic quadrilateral, and so $\angle DCM = \angle MBE = \angle CAM$, therefore triangles AMC and CMD are similar, hence $\frac{|AC|}{|CD|} = \frac{|MC|}{|MD|}$. Similarly, $\frac{|AB|}{|BD|} = \frac{|MB|}{|MD|}$, and as $|MB| = |MC|$, the two ratios are equal, whence the desired relation. This may be also proven using equalities of areas.

Problem.2. Prove that, in any convex quadrilateral, (at least) one diagonal is longer than one-fourth of the perimeter.

Solution. Consider convex quadrilateral $ABCD$ with diagonals AC, BD meeting at O. By repeated applications of the triangle inequality for the four small triangles created, we get

$$|AO| + |OB| > |AB|, \quad |BO| + |OC| > |BC|,$$

$$|CO| + |OD| > |CD|, \quad |DO| + |OA| > |DA|,$$

whence by summing them we obtain $2(|AC| + |BD|) >$ perimeter of $ABCD$. If both diagonals were not longer than one-fourth of the perimeter it would contradict the above inequality.

Problem.3. Let n be a positive integer. Find (with proof) all positive integer solutions for the equation

$$7 \cdot 4^n = a^2 + b^2 + c^2 + d^2.$$

Solution. Everything is based on the fact that perfect squares yield remainder 0 or 1 when divided by 4. Therefore if at least one of a, b, c, d is odd, all must be odd. Now, it is also true that an odd number $2m + 1$, when squared, gives remainder 1 when divided by 8, as $(2m + 1)^2 = 4m(m + 1) + 1$ and $m(m+1)$ is always even. Therefore this offers no solution when $n > 1$ as then $7 \cdot 4^n$ is divisible by 8, while $a^2 + b^2 + c^2 + d^2$ is not, and for $n = 1$ it offers the only solutions (up to a permutation) $7 \cdot 4 = 28 = 1^2 + 1^2 + 1^2 + 5^2$ and $7 \cdot 4 = 28 = 1^2 + 3^2 + 3^2 + 3^2$.
If all a, b, c, d are even we can simplify by 4 to get a similar equation, only that now we may also consider the case $n = 0$, which yields the unique solution $7 \cdot 1 = 7 = 1^2 + 1^2 + 1^2 + 2^2$.
In conclusion, all possible solutions are (up to a permutation)
$\{2^{n-1}, 2^{n-1}, 2^{n-1}, 5{\cdot}2^{n-1}\}$, $\{2^{n-1}, 3{\cdot}2^{n-1}, 3{\cdot}2^{n-1}, 3{\cdot}2^{n-1}\}$ and $\{2^n, 2^n, 2^n, 2^{n+1}\}$.

Problem.4. If p, q are prime numbers, and $\frac{p}{x} + \frac{q}{y}$ is an integer, where $x < p$, $y < q$ are positive integers, then show that it follows with necessity that $x = y$.

Solution. The condition is equivalent to $py + qx$ being divisible by xy. But then x must divide py, and as $x < p$ and p is prime, it follows that x must divide y. Similarly we obtain that y must divide x. As both x, y are positive, this implies $x = y$.

Problem.5. Prove that, for any positive integer $n \geq 5$, the set $X_n = \{1, 2, \ldots, n\}$ can be partitioned into two subsets S_n and P_n, such that the sum of the elements of S_n be equal to the product of the elements of P_n.

Solution. The naive attempt of trying $P_n = \{p\}$ (one element only) fails miserably, as $(1 + 2 + \cdots + n) - p = n(n+1)/2 - p > p$ for any $1 \leq p \leq n$.
The (less) naive attempt of trying $P_n = \{p,\, q\}$ (two elements only) fails for more subtle reasons, as $(1 + 2 + \cdots + n) - p - q = n(n+1)/2 - p - q = pq$ is equivalent to $n(n+1)/2 + 1 = (p+1)(q+1)$, which does not always have a solution in positive integers, with $1 \leq p, q \leq n$.
Trying to find "by hand" solutions for small values of n yields $P_5 = \{1, 2, 4\}$, $P_6 = \{1, 2, 6\}$, $P_7 = \{1, 3, 6\}$, $P_8 = \{1, 3, 8\}$. This encourages us to try $P_n = \{1, p, q\}$, which after similar to the above calculations leads to $n(n+1)/2 = (p+1)(q+1)$. Now, for n even we can take $P_n = \{1, (n-2)/2, n\}$, while for n odd we can take $P_n = \{1, (n-1)/2, (n-1)\}$.

Solutions of the Selected Problems of IMO Training Camp Tests

Here we give solutions of the selected problems of IMO training camp tests together with solutions of the problems given in the qualifying tests for the Pakistani National Team participating in IMO.

First Test:

Problem.1. Let ABC be a triangle. Show that $\angle BAC = 90^0$, if and only if

$$\cos B + \cos C = \frac{b+c}{a}$$

Solution. Assume that $\hat{A} = 90^0$. Then $\triangle ABC$ is right angled in $\hat{A}$ and $\cos B = \frac{c}{a}$, $\cos C = \frac{b}{a}$. By adding these equalities one obtains the required equality.

Assume that $\cos B + \cos C = \frac{b+c}{a}$. By cosine law we have:

$$\cos B = \frac{a^2+c^2-b^2}{2ac} \quad \text{and} \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

By adding these two equalities and equating with $\frac{b+c}{a}$ one obtains:

$$\frac{a^2+b^2-c^2}{2ab} + \frac{a^2+c^2-b^2}{2ac} = \frac{(b+c)(a-b+c)(a+b-c)}{2abc} = \frac{b+c}{a}$$

$$\Rightarrow (a-b+c)(a+b-c) = 2bc \quad \Rightarrow \quad a^2 = b^2 + c^2$$

Problem.2. Let a, b, c be positive numbers such that

$$\frac{a(b-c)}{b+c} + \frac{b(c-a)}{c+a} + \frac{c(a-b)}{a+b} = 0$$

show that $(a-b)(b-c)(c-a) = 0$.

Solution. The first step is to eliminate the denominators. One obtains

$$a(b-c)(c+a)(a+b) + b(c-a)(b+c)(a+b) + c(a-b)(b+c)(c+a) = 0$$

It is convenient to use cyclique summation:

$$\sum a(b-c)(a^2 + ab + bc + ca) = 0$$

So, by grouping that part which contains $\sum ab = ab + bc + ca$, One obtains:

$$\sum ab \cdot \sum a(b-c) + \sum a^3(b-c) = 0.$$

Since $\sum(ab - ac) = 0$, we have only $\sum a^3(b-c) = 0$. An easy computation shows that

$$\sum a^3(b-c) = (a+b+c)(-a+b)(-b+c)(-c+a) = 0.$$

The result follows.

Problem.3. In a group of 3 persons, the average of the age of any 2 persons is the age of the third person. Show that the total age of the three persons is divisible by 3.

Solution. If the ages are x, y, z, then $x + z = 2y$ and $y + z = 2x$. By subtracting the equations one has $x - y = 2(y - x)$, which gives $x = y$. Analogously, $y = z$, so $x + y + z = 3x$ is a multiple of 3.

Second Test:

Problem.1. Let a, b, c be positive integers such that $\frac{a+b}{bc} = \frac{b+c}{ca} = \frac{c+a}{ab}$ show that $a = b = c$.

Solution. Since $\frac{a+b}{bc} = \frac{b+c}{ca} = \frac{c+a}{ab}$. Multiply by abc, we get $a(a+b) = b(b+c) = c(c+a)$.

Let $d = \gcd(a,b,c)$ and $a = dA$, $b = dB$, $c = dC$. (as $d|a \Rightarrow a = dA$, $d|b \Rightarrow b = dB$, $d|c \Rightarrow c = dC$)

Now $a(a+b) = b(b+c) = c(c+a)$. Putting values of a, b and c from above, we get

$$dA(dA+dB) = dB(dB+dC) = dC(dC+dA)$$

$$d^2A(A+B) = d^2B(B+C) = d^2C(c+A)$$

dividing by d^2 throughout, we get

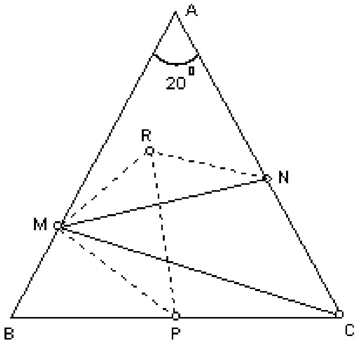
$$A(A+B) = B(B+C) = C(C+A)$$

where $\gcd(A,B,C) = 1$.

If p is prime divisor of A, it follows that $p|B$ and $p|C$ (If $p|A \Rightarrow p|A(A+B) \Rightarrow p|C(C+A) \Rightarrow p|C$ or $p|C+A \Rightarrow p|C$. Similarly $p|B$.) So $p|A, B, C$. So $A = B = C = 1$. Then $a = b = c$.

Problem.2. We are given an isosceles triangle ABC such that $AB = AC$ and $\angle BAC = 20^0$. Let M be the foot of the altitude from C and N be a point on the side AC such that $CN = \frac{1}{2}BC$. Find the size of the angle $\angle AMN$.

Solution.



Let P be the mid-point of BC. Then MP is the median line in $\triangle BMC$. It follows that $PB = PM = PC = CN$ (as $CN = \frac{1}{2}BC = PC = PB$ (given)), take the point R such that $PCNR$ is parallelogram (infact it is a rhombus).

Then

$$\angle R\hat{P}M = \angle R\hat{P}B - \angle M\hat{P}B = \angle A\hat{C}B - (180^0) - 2\angle M\hat{B}C =$$

$$= \angle A\hat{C}B + 2\angle M\hat{B}C - 180^0 = 80^0 + 2(80^0) - 180^0 = 60^0$$

and $RP = MP$. That is $\triangle MPR$ is equilateral.

Hence $MR = RP = RN$ and $\angle M\hat{R}N = \angle M\hat{R}P + \angle P\hat{R}N = 60^0 + 80^0 = 140^0$.

Then $\angle R\hat{M}N = \angle R\hat{N}M = 20^0$ and $\angle A\hat{N}M = 20^0 + 80^0 = 100^0$. Follows $\angle A\hat{M}N = 180^0 - (100^0 + 20^0) = 180^0 - 120^0 = 60^0$. Thus $\angle A\hat{M}N = 60^0$.

Problem.3. Find the least value of the sum

$$|x-1| + |x-2| + \text{........} + |x-10|$$

when x runs over the set of all real numbers.

Solution. Let $E = |x-1| + |x-2| + \text{........} + |x-10|$. We have $|x-k| + |x-(11-k)| \geq 11-2k$, where $[|11-2k| = |11-k-k+x-x| = |x-k-(x-11+k)| \leq |x-k| + |-(x-11+k)| = |x-k| + |x-11+k|]$, and equality occurs when $x \in [k, 11-k]$. By adding these equalities for $k = 1, \ldots\ldots, 5$, $E \geq 55 - 2(a+2+3+4+5) = 55-30 = 25$. $\Rightarrow$ $E \geq 25$.

Equality occurs when

$$x \in \cap_{1 \leq k \leq 5}[k, 11-k] = [5, 6].$$

Minimum value $= 25$.

First Selection Test for IMO (January 25, 2007)

Time allowed: $3\frac{1}{2}$ Hours

Problem.1. Let ABC be a triangle and D be the point where the angle bisector of $\angle A$ meets BC. Prove that the line BC, the perpendicular bisector of AD and the perpendicular at A on AC are concurrent if and only if $\angle B$ is a right angle.

Solution. Let E be the meeting point of BC and the perpendicular bisector of AD. Let $\angle BAD = \angle CAD = \frac{1}{2}\angle A = \alpha$ and $\angle BAE = \beta$. Then triangle AED is isosceles, so $\angle ADE = \angle DAE = \alpha + \beta$; but $\angle ADE = \alpha + \angle C$, therefore $\angle C = \beta$. Now, $\angle EAC = \beta + \angle A = \angle C + \angle A = \pi - \angle B$, so AE perpendicular on AC iff $\angle B$ is a right angle.

Notice that cases $\angle B \leq \angle C$ and $\angle A$ right angle would lead to impossible constructions, hence these pathological cases are also accounted for.

Problem.2. Prove the following inequality

$$(ab + bc + ca)(1 + 18abc) \geq 15abc$$

for $a\,, b\,, c$ non-negative real numbers with $a + b + c = 1$.
State, with proof, the case(s) of equality.

Solution. We have $ab + bc + ca \geq \sqrt{3}\sqrt{abc}$, by Newton's inequality. Denote $x = \sqrt{abc}$, then it is enough to prove $\sqrt{3}x(1 + 18x^2) \geq 15x^2$, or $18\sqrt{3}x^2 - 15x + \sqrt{3} \geq 0$. The roots of this quadratic are $\frac{2}{3\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$, but $x = \sqrt{abc} \leq \sqrt{(\frac{a+b+c}{3})^3} = \frac{1}{3\sqrt{3}} < \frac{2}{3\sqrt{3}} < \frac{1}{\sqrt{3}}$, so the inequality is proven.

The equality cases occur for $a = b = c = \frac{1}{3}$, as well as for two of a, b, c equal to 0 and the third equal to 1, as seen from the above calculations.

Alternatively, one may homogenize the inequality, perform all calculations (using brute force), and finally apply Muirhead's inequality.

Notice that the naive approach of immediately using AM-GM inequality

$$\Big(\frac{ab + bc + ca}{3}\Big)^3 \geq (abc)^2$$

will miserably fail.

Problem.3. Solve in non-negative integer numbers

$$x^2 + y^4 + z^6 = 2^{1111}$$

Solution. It is advantageous to use the notations $x = a$, $y^2 = b$, $z^3 = c$ with which $a^2 + b^2 + c^2 = 2^{1111}$. Now, $\gcd(a, b, c) | 2^{1111}$ so take $\gcd(a, b, c) = 2^k$ with $0 \leq k \leq \frac{1111}{2}$, so for $a = 2^k u$, $b = 2^k v$, $c = 2^k w$ with $\gcd(u, v, w) = 1$ we get $u^2 + v^2 + w^2 = 2^{1111-2k}$. This has no solution for $1111 - 2k > 1$, as quadratic residues modulo 4 are only 0 and 1, and u, v, w cannot be simultaneously even. Therefore $2k = 1110$, with only solutions for two of u, v, w being 1 and the third 0, but divisibility to 2 and 3 constraints only allow the solution $x = 2^{555}$, $y = 0$, $z = 2^{185}$.

Problem.4. In a cricket league, n teams play a complete round-robin tournament, with 2 points awarded for a win, 1 point for a tie and 0 points for a loss. Determine the admissible n such that the tournament could end with each team having won exactly half of its points against the last 6 ranking teams (this being true for these teams also, each of them having won exactly half of its points against the other 5).

Solution. Let us call the last 6 ranking teams *the losers*, and the other $n-6$ teams *the winners*. It is clear that the losers together have won $2(2C_2^6)$ points, while the winners together have won $2(2(C_2^{n-6})$ points; but in all there are $2C_2^n$ points awarded, hence we have the equation $2(2C_2^6)+2(2C_2^{n-6}) = 2C_2^n$ which yields $n^2-25n+144=0$, so we need have $n=16$ or $n=9$. But atleast one loser won not less than 10 points, while at least one winner has won at most $2(n-7)$ points; this implies $n \geq 12$, therefore $n=16$ and $n-6=10$.

We will give an example of high symmetry: assume all losers tied the matches between themselves, and similarly, all winners tied the matches between themselves; this gives that each loser won 10 points, while each winner won 18 points. Arrange the winners in a rectangular array of 2 rows and 5 columns; there will be two groups of one row each. Establish a one-to-one mapping between a partition of the losers in 2 classes of 3 losers each and these groups, and have each loser tie the winners in the corresponding group, and lose to the others. Thus, each loser ties 5 and loses the other 5 of these cross-matches, accounting for 5 points, while each winner ties 3 and wins the other 3 of these cross-matches, accounting for 9 points.

Second Selection Test for IMO (January 26, 2007)

Time allowed: $3\frac{1}{2}$ **Hours**

Problem.1. Triangle ABC is isosceles with $CA = CB$, I is its incenter and M is the meeting point of CI and AB.

1. Show that the circumcircle Γ of triangle AIB is tangent to AC, BC;
2. Show that for any point P on the arc AIB of Γ the following holds

$$\frac{PC}{PM} = \frac{AC+BC}{AB};$$

3. Do the results from points 1. and/or 2. remain true for a scalene triangle ABC?

Solution. 1. It has been proven that the center ω of Γ lies on CI. As $A\omega I$ is isosceles, $\angle\omega AI = \angle\omega IA$ and so $\angle\omega AC = \angle\omega IA + \angle IAC = \angle\omega IA + \angle IAM = \frac{1}{2}\pi$, therefore AC is tangent to Γ (and similarly BC is tangent to Γ).

2. The bisector theorm in triangle CAM yields $\frac{IC}{IM} = \frac{AC}{AM}$ (and similarly $\frac{IC}{IM} = \frac{BC}{BM}$), therefore Γ is an Apollonius circle for points C and M, so $\frac{PC}{PM}$ will be constant for all points P on the arc AIB of Γ, and the constant value is $\frac{AC+BC}{AB}$.

3. According to the above, point 2. will stay true for a scalene triangle, but point 1. will clearly no longer stay true.

Problem.2. Let $P \in \mathbb{Z}[X]$ be a polynomial of degree $d \geq 2$. Prove that the sequence $(a_n)_{n\geq 1}$ given by $a_n = P(n)$ may contain d (but cannot contain more than d) successive terms in arithmetical progression.

Solution. One can exhibit (possibly by invoking Lagrange interpolation polynomials)

$$P(x) = (x-1)(x-2)......(x-d) + r(x-1) + a$$

with $\deg P = d$ and $P(k) = a + (k-1)r$ for $1 \leq k \leq d$.

If $P(k_0 + k) = a + kr$ for more than d successive values $k = 0, 1, ...$, then

$$Q(x) = P(x+1) - P(x) - r = 0$$

for at least d values of x; but $\deg Q = d - 1$, so $Q = 0$. However, this would contradict $d \geq 2$.

Problem.3. Given integer numbers x_s, $1 \leq s \leq n$, of distinct remainders when divided by n, determine the remainder of division by $n \geq 3$ for the expression

$$E = \sum_{1\leq i<j<k\leq n} x_i x_j x_k$$

Solution. It is clear that we may replace each x_s with its remainder when divided by n, therefore the set of integers x_s may be replaced with the set of values $1 \leq s \leq n$. The simplest way to compute now is to use the identity

$$\Big(\sum_{i=1}^{n} i\Big)^3 = \sum_{i=1}^{n} i^3 + 3 \sum_{1\leq i\leq n} \Big[i^2\big(\big(\sum_{1\leq j\leq n} j\big) - i\big)\Big] + 6 \sum_{1\leq i<j<k\leq n} ijk$$

The above is thus equivalent to

$$6E + P_1^3 - 3P_1P_2 + 2P_3$$

where

$$P_1 = \sum_{1\leq i\leq n} i = \frac{n(n+1)}{2}, \quad P_2 = \sum_{1\leq i\leq n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$P_3 = \sum_{1\le i\le n} i^3 = \frac{n^2(n+1)^2}{4}$$

and simple calculations yield the value

$$6E = \frac{n^2(n+1)^2(n-1)(n-2)}{8}.$$

Now it is easy to see that $E \equiv 0 \pmod{n}$ for $n \equiv 0, 1, 2, 3, 5, 7 \pmod{8}$, while $E \equiv \frac{n}{2} \pmod{n}$ for $n \equiv 4, 6 \pmod{8}$.

Problem.4. Determine the values n for which a regular polygon with n sides (n-gon) can be triangulated by non-crossing diagonals in such a way that all triangles in the triangulation be isosceles.

Solution. We claim the answer is $n = 2^a(2^b + 1)$, with a, b non-negative integers, $a + b \ge 1$. Clearly, each side l of the $n-$gon will belong to (exactly) one triangle $\triangle$ in the triangulation. If the other two sides of $\triangle$ were diagonals, they should be equal in length, so the apex of $\triangle$ should be on the perpendicular bisector of l. This cannot be for n even, and is possible once only for n odd (otherwise diagonals will cross).

Therefore for even n the only possibility is for the sides of the $n-$gon to belong, in pairs of consecutive sides, to isosceles triangles making a "bracelet" that leaves within a regular polygon with $\frac{n}{2}$ sides. Iteration of this mandatory procedure reaches a regular polygon with odd number of sides (or the procedure ends when n is a power of 2). For a regular polygon with odd number of sides, a triangle $\triangle$ having two of its sides equal length diagonals will necessarily occur (once and only once), as described in the above. The two symmetrical regions, on one side and the other of $\triangle$, will be suitable for triangulation with isosceles triangles only with triangles having, as equal sides, consecutive sides of the polygon; as the iteration of this procedure will have to exhaust the sides of the (intermediately created) polygon, it follows that the number of the sides of the $n-$gon will have to be equal to 1+ a power of 2, wherefore the justification of our starting claim.

Let us notice that for $b = 1$ (and only then), one of the isosceles triangles will in fact be equilateral.

Third Selection Test for IMO (April 27, 2007)

Time allowed: $3\frac{1}{2}$ Hours

Problem.1. Show that, for $n \geq 4$, the equation

$$1 = \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}$$

has at least two sets of integral solutions, satisfying $1 < x_1 < x_2 < ...... < x_n$.

Solution. One starts with the identity

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

and keeps using the "splitting identity"

$$\frac{1}{m} = \frac{1}{m+1} + \frac{1}{m(m+1)}$$

repeatedly for the fraction with largest denominator, until reaching n terms.

Alternatively, one has the identity

$$1 = \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}}$$

where the last term can be splited by

$$\frac{1}{2^{n-2}} = \frac{1}{3 \cdot 2^{n-3}} + \frac{1}{3 \cdot 2^{n-2}}$$

thus getting a n terms expression.

Problem.2. Let l_1, l_2 be two lines in the plane and M a variable point on the line l_1. One considers the circle Γ of center M which is tangent to l_2 and the circle Γ_1 tangent to l_1, l_2 and Γ. Let T be the touching point of Γ and Γ_1 . Find the locus of T when M varies on l_1.

Solution. The locus contains line l_2. If l_1 and l_2 are parallel, the locus contains another line l, parallel to l_1 and l_2 such that $\frac{d_1}{d_2} = \frac{1}{2}$, where d_1, d_2 are the distances from M to l_1, l_2. If $L_! \cap l_2 = \{O\}$, then the locus contains also two lines through O, the locus of points satisfying $\frac{d_1}{d_2} = \frac{1}{2}$. In this case we have to delete the point O itself (corresponding to degenerate circles).

Problem.3. There are 121 people at a party. Show that either there are 13 people who mutually do not know each other, or there is a person who knows at least 10 others. Find and prove a general statement.

Solution. A general statement could be: "In a graph with $(k-1)n+1$ vertices, there is a k subgraph with no edge, or there is a vertex of degree at least n". Let us suppose that all degrees are less than n. Choose a vertex V_1 and consider all the vertices connected with V_1, and consider A_1 the set of these vertices. Next choose a vertex V_2 in the complement of A_1 and consider all the vertices connected with V_2, and let A_2 be the set of these vertices not contained in A_1, and so on. Because $A_1, A_2, \ldots.$ have cardinalities at most n, then we can choose V_{k-1} and the corresponding set A_{k-1}, and the cardinality of $A_1 \cup A_2 \cup \ldots. \cup A_{k-1}$ is atmost $(k-1)n$. Hence we can choose a vertex V_k in the complement of this union and the subgraph $\{V_1, \ldots., V_k\}$ has no edge.

Fourth Selection Test for IMO (April 28, 2007)

Time allowed: $3\frac{1}{2}$ Hours

Problem.1. A sequence of real numbers $1, a_0, a_1, a_2, \ldots....$ is defined by the formula

$$a_{n+1} = \lfloor a_n \rfloor \cdot \langle a_n \rangle$$

for $n \geq 0$; here a_0 is an arbitrary real number, $\lfloor a_n \rfloor$ denotes the greatest integer not exceeding a_n, and $\langle a_n \rangle = a_n - \lfloor a_n \rfloor$. Prove that $a_n = a_{n+2}$ for n sufficiently large.

Hint. Case 1: $a_0 \geq 0$. In this case the sequence $\lfloor a_n \rfloor$ is strictly decreasing until becomes constant 0, so the sequence is eventually constant.

Case 2: $a_0 < 0$. In this case the sequence $\lfloor a_n \rfloor$ is increasing, therefore there is an integer k such that $\lfloor a_n \rfloor = k$ for large n. If $k = 0$, the sequence becomes constant. Otherwise, studying the function $f(x) = k(x-k)$ on the interval $(k, k+1)$ we find that, for a large n, $a_n = \frac{k^2}{k-1}$ (constant) or $a_n = a, \;\; a_{n+1} = -a-1$, for some a in $(-1, 0)$.

Problem.2. A convex polygon $\mathcal{P}$ is *decomposable* if there is a point O and two other convex polygons $\mathcal{Q}$ and $\mathcal{R}$ such that any point A in $\mathcal{P}$ can be written as

$$\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}$$

where B is in $\mathcal{Q}$ and C is in $\mathcal{R}$ and no side of $\mathcal{Q}$ is parallel to a side of $\mathcal{R}$. Find all regular polygons which are decomposable.

Hint. Take $\mathcal{Q}$ a regular polygon with $2n$ sides, and $\mathcal{R}$ the same polygon rotated with $\frac{\pi}{2n}$; their sum will be a regular polygon with $4n$ sides.

Problem.3. Let $P(X) \in \mathbb{Z}[X]$ a polynomial of degree $n \geq 4$ and consider the image set $A = \{P(k) \setminus k \in \mathbb{Z}\}$. If $\{0, 1, 2, ..., n-1\}$ is a subset of A, prove that $\{n, n+1, ..., 2n-1\}$ is disjoint from A.

Hint. If $P(a_k) = k$, we can change $P(X)$ with $P(X + a_0)$, so we can suppose that $P(0) = 0$. Next we can show that $a_k = \pm k$, so $P(X) = cX(x-1).......(X-n+1)+X$ or $P(X) = cX(X+1).........(X+n-1)+X$. We use now the inequality $n! - 1 > 2n - 1$.

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