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# NATIONAL MATHEMATICS OLYMPIAD SELECTED PROBLEMS (AUGUST 2004 - APRIL 2006)

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## 1. Introduction

The main mathematical competition in Pakistan at high school level is the National Mathematics Olympiad (NMO). It was first held in August 2004 and is conducted on annual basis. A large number of young students take part in this competition every year since. It is organized in several rounds.
The School of Mathematical Sciences (SMS) is the founder of the NMO. To encouraging excellence in the educational environment in Pakistan, the School of Mathematical Sciences has taken the initiative to provide a world class coaching to the young Pakistani students participating in the different mathematics contests at National and International level.
This collection consists of all the selected problems which appeared in different rounds of the NMO together with the problems given in the qualifying tests for the Pakistani National Team participating in the International Mathematical Olympiad (IMO).
This collection is intended as practice for the serious student who wishes to improve her or his performance in National and International contests.We do hope that mathematics teachers will also be involved to come up with new original problems.
The collection contains non trivial material, and one should be prepared to work with pencil and paper at hand and to return to difficult places more than once.

## 2. The IMO Training Camp

### 2.1. First round (Auguest 22 - 28, 2004)

**Time allowed: 3 Hours.**

**Problem.1.** Let $(1+\sqrt{2})^n = A_n + B_n\sqrt{2}$.

Where $n \in N$ and $A_n, B_n \in Q$

Find $\lim\limits_{n\to\infty} \frac{A_n}{B_n}$.

**Problem.2.** Prove that $(p+1)$ divides $C_p^{2p}$ for all $p \in N$.

**Problem.3.** Let $a, b, c$ be positive reals. First show that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$

then generalize the above inequality and give a proof of it.

### 2.2. First selection test for IMO (November 30, 2004)

**Time allowed: 3 Hours.**

**Problem.1.** Find all positive integers $n$ with the following property:
There exist integers $a$ and $b$ such that $n^2 = a + b$ and $a^3 = a^2 + b^2$.

**Problem.2.** Let $a, b, c$ be the lengths of the sides of a triangle $ABC$.
a) Prove that there exists a triangle whose sides have lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$.
b) Show that

$$1 \leq \frac{a+b+c}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} < 2$$

**Problem.3.** Let $ABC$ be a right angle triangle such that $\angle A = 90^\circ$ and $\angle B$ is greater than $\angle C$. An arbitrary point $D$ is taken on the segment $BC$. The angle bisector of $\angle ADB$ intersects $AB$ in $M$ and the angle bisector of $\angle ADC$ intersects $AC$ in $N$. The line $MN$ intersects $BC$ in $P$. Show that $\angle MPB = \frac{1}{2}(\angle B - \angle C)$ iff $D$ is the foot of the altitude from $A$.

## 2.3. Final selection test for IMO (December 02, 2004)

**Time allowed: 3 Hours.**

**Problem.1.** In a given triangle $ABC; O$ is the circumcenter; $D$ is the midpoint of the side $AB$ and $E$ is the centroid of the triangle $ACD$.
a) Show that if $AB = AC$, then the lines $CD$ and $OE$ are perpendicular.
b) Show that if $CD$ and $OE$ are perpendicular, then $AB = AC$.

**Problem.2.** We are given a rectangle $ABCD$ whose sides have lengths $AB = CD = a$, $BC = AD = b$ and $a < b$. A point $P$ is moving along the perimeter of the rectangle. Find the positions of $P$ for which the sum $PA + PB + PC + PD$ is least.

**Problem.3.** Let $n$ be a positive integer which is divisible by 7. Show that the number $N = (n+1)(n+2)(n+3)(n+4)(n+5)(n+6)$ cannot be a perfect square.

**Problem.4.** Let $a, b, c$ be non negative real numbers which are not greater than 1 and such that $ab + bc + ca = 1$. Show that $a^2 + b^2 + c^2 \leq 2$.

# 3. The First National Mathematics Olympiad

## 3.1. Tests for NMO at different cities of Pakistan
## First Stage:

### 3.1.1. *__Islamabad Center__*.

**Time allowed: 1.5 Hours**

**Problem.1.** Let $x$ be a real number such that $x^4 + \frac{1}{x^4} = 2$. Find the possible values of $x + \frac{1}{x}$.

**Problem.2.** How many 4-digit numbers are there which begin with 1 and have exactly two identical digits? (Examples are $1557, 1030, 1321$.)

**Problem.3.** Let $P$ be a point in the interior of the triangle $ABC$. The reflections of $P$ across the midpoints of the sides $BC, CA, AB$, are $P_A, P_B, and\ P_C$ respectively. Prove that the lines $AP_A, BP_B and\ CP_C$ are concurrent.

### 3.1.2. *__Karachi Center__*.

**Time allowed: 1.5 Hours**

**Problem.1.** Find all positive integers $n$ such that the number $n^8 + n^4 + 1$ is a prime.

**Problem.2.** Let $n \geq 2$ be an integer and let $x_1, x_2, ......., x_n \in \{-1, 1\}$ such that $x_1x_2 + x_2x_3 + ....... + x_{n-1}x_n + x_nx_1 = 0$. Prove that $n$ is divisible by 4.

**Problem.3.** Let $ABCD$ be a convex quadrilateral. Prove that there exists a circle in its interior touching all its sides if and only if $AB+CD=AD+BC$.

3.1.3. ***Lahore Center.***

**Time allowed: 1.5 Hours**

**Problem.1.** Let $m, n$ be positive integers such that $\frac{7}{10} < \frac{m}{n} < \frac{11}{15}$. Find the smallest possible value of $n$.

**Problem.2.** Denote by $P$ the perimeter of triangle $ABC$. If $M$ is a point in the interior of the triangle, prove that

$$\frac{1}{2}P < MA + MB + MC < P.$$

**Problem.3.** Prove that if an arithmetical progression of positive integers contains a square, then it contains infinitely many squares.

3.1.4. ***Peshawar Centers.***

**Time allowed: 1.5 Hours**

**Problem.1.** Let $a, b, c$ be real numbers. Prove the inequalities:
a) $a^2 + b^2 + c^2 \geq ab + bc + ca$;
b) $a^4 + b^4 + c^4 \geq abc(a + b + c)$.

**Problem.2.** Prove that the product of four consecutive positive integers:
a) is divisible by 24;
b) is never a perfect square.

**Problem.3.** The side length of the equilateral triangle $ABC$ equals $l$. The point $P$ lies in the interior of $ABC$ and the distances from $P$ to the triangle's sides are $1, 2, 3$. Find the possible values of $l$.

3.1.5. ***Skardu Center.***

**Time allowed: 1.5 Hours**

**Problem.1.** How many solutions has the equation $A \cup B = \{1, 2, 3, 4\}$? (two solutions $(A_1, B_1)$ and $(A_2, B_2)$ are different iff $A_1 \neq A_2$ or $B_1 \neq B_2$; for instance, $A_1 = \{1, 3, 4\}$, $B_1 = \{1, 2, 4\}$ and $A_2 = \{1, 2, 4\}$, $B_2 = \{1, 3, 4\}$ are different solutions.)

**Problem.2.** Can you find a polynomial $f \in Z[X]$ such that $f(Z) \subset N$ and $f$ takes the value $-2005$?

**Problem.3.** Compute $cos\frac{\pi}{5}$.

## 3.2. First Training Camp (Augest 17 - 23, 2005).

### 3.2.1. *First Test*.

**Time allowed: 2 Hours**

**Problem.1.** Given any five integers, show that there is at least one subset of three integers whose sum is divisible by 3, e.g. $\{9, -13, 1\}$ whose sum is divisible by 3.

**Problem.2.** Find the value of

$$\sqrt{1-\sqrt{\frac{17}{16}-\sqrt{1-\sqrt{\frac{17}{16}-\sqrt{1-\ldots\ldots}}}}}$$

Assuming that it is a real number.

**Problem.3.** The radius of the circumcircle of an isosceles triangle is $R$ and the radius of its inscribed circle is $r$. Prove that the distance between the two centers is $\sqrt{R(R-2r)}$.

**Problem.4.** Find all the positive integers $n$ such that the number $2^4+2^7+2^n$ is a perfect Square.

**Problem.5.** Prove the following equality:

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\ldots\ldots\ldots\ldots+\sqrt{1+\frac{1}{2004^2}+\frac{1}{2005^2}}=2005-\frac{1}{2005}$$

## 3.3. First Training Camp(Augest 17 - 23, 2005).

### 3.3.1. *Final Test*.

**Time allowed: 3 Hours**

**Problem.1.** Let $a, b, c$ be the positive numbers such that $ab + bc + ca = 1$. Show that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\sqrt{3}+\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}$$

**Problem.2.** A sequence $(a_n)_{n\in N}$ is given by

$$a_1 = 1;\quad a_n = \frac{4n-2}{n}a_{n-1};\quad n \geq 2$$

Prove that all the terms of the sequence are positive integers.

**Problem.3.** Consider the two circles in the diagram intersecting at points $A$ and $D$. Let $ARC$ and $BPD$ be the straight lines. $DR$ extended meets $BC$ at point $S$. Prove that $PR$ is parallel to $BC$. ($BC$ is not tangent to the smaller circle).

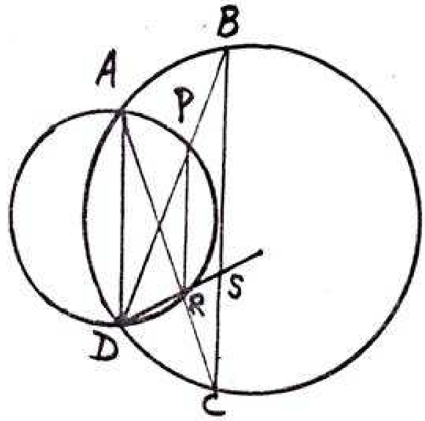


**Figure No. 1**

3.4. **First selection test for IMO (January 24, 2006)**

**Time allowed: 3 Hours.**

**Problem.1.** The circle inscribed in the triangle $ABC$ touches the sides $BC, CA, AB$ at points $A'$,$B'$, $C'$, respectively. Prove that the lines $AA'$, $BB'$, $CC'$ are concurrent.

**Problem.2.** Show that the equation $x^2 + x + 2 = 20y^3$ has no integral solutions.

**Problem.3.** Let $ABCD$ be a convex quadrilateral. Consider the points $A'$ and $B'$ such that $C$ is the midpoint of the line segment $AA'$ and $D$ is the midpoint of the line segment $BB'$. Evaluate with proof, $\frac{[ABA'B']}{[ABCD]}$ .

**Problem.4.** Let $a, b, c$ be positive real numbers. Prove that

$$\frac{1}{2} \leq \frac{a}{a+2b+3c} + \frac{b}{b+2c+3a} + \frac{c}{c+2a+3b} < 1.$$

**Problem.5.** Find, with proof, the greatest number of elements of a subset $M$ of the set $A = \{1, 2, ......, 2006\}$ having the following property:
The sum of any two distinct elements of $M$ is not divisible by 12.

### 3.5. **Second selection test for IMO (January 30, 2006)**

**Time allowed: 3 Hours.**

**Problem.1.** Let $ABCD$ be a convex quadrilateral, such that $BC = CD$ and $\angle BCD = 90°$. Prove that $\frac{AB+AD}{AC} = \sqrt{2}$. if and only if $\angle BAD = 90°$ .

**Problem.2.** a) Find two numbers $a, b \in N$ such that $b$ is not divisible by $a$ and, for any $k = 1, 2, ........, 10$; $a^k$ divides $b^{k+1}$ (*)
b) What can you say if (*) is true for infinitely many $k$?

**Problem.3.** Let $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{5}$ and $x_{n+2} = \frac{x_{n+1}x_n}{5x_n - 6x_{n+1}}, \forall n \geq 1$. Find a closed formula for $x_n$.

**Problem.4.** 2006 points are chosen inside a circle having area 1000 such that no three of them are collinear. Prove that one can choose three points which are the vertices of a triangle with area less than 0.499.

### 3.6. **Third selection test for IMO (January 31, 2006)**

**Time allowed: 3 Hours.**

**Problem.1.** Let $P$ be a point inside the triangle $ABC$ and let $P_a, P_b, P_c$ be its projections onto the sides $BC, CA, AB$, respectively. Prove that

$$PP_a^2 + PP_b^2 + PP_c^2 \geq \frac{4S^2}{a^2 + b^2 + c^2}.$$

where $S$ denotes the area of triangle $ABC$.

**Problem.2.** Solve the equation $x^2 + 2x = 5y^2 + 4$ with integral $x$ and $y$.

**Problem.3.** Let $a_\circ = a_1 = 1$ and $a_{n+1} = \frac{a_n(a_n+1)}{a_{n-1}}$, $\forall n \geq 1$. Prove that all terms of the sequence are positive integers.

**Problem.4.** In a class with 20 children, each of them has at least 10 friends. Prove that there exist two groups of 3 children such that every child of one group is a friend of any child of the second group.

### 3.7. **Fourth selection test for IMO (April 09, 2006)**

**Time allowed: 3 Hours.**

**Problem.1.** a) Find all polynomial functions $P$ verifying for all $x \in R$

$$P(x) + 1 = \frac{[P(x+1) + P(x-1)]}{2}.$$

b) Are there non polynomial functions $f : R \to R$, which satisfy the same equation?

**Problem.2.** Consider a connected graph such that among its vertices there are the vertices $A, B, C, D$ of a given square and the sum of the lengths of its edges is minimal.

**Problem.3.** We are given a $n \times k$ rectangle which is divided into $nk$ unit cells distributed in $n$ rows and $k$ columns. How many ways can we assign either $+1$ or $-1$ in each cell such that the following conditions are satisfied:
a) The product of all elements of a row equals 1,
b) The product of all elements of a column equals $-1$ ?

**Problem.4.** Find all integer values of $k \geq 1$ satisfying the following:
There exists an odd number $a$ such that all the remainders (mod $2^k$ ) of $a, a^2, a^3, ........, a^b$ (where $b = 2^{k-1}$) are different.

The SMS invites solutions to these problems and also new original problems from mathematics teachers and students. Please send your solutions and problems to: saifullahkhalid75@yahoo.com. Best solutions with the names of the solvers will be published in the next issue of the MATH TRACK and will also be posted on the website: www.sms.edu.pk .

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