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MEASURABLE SPACE FRAMES

O. I. REINOV

Saint Petersburg State University, Saint Petersburg
R u s s i a
Email: orein@or1146.spb.edu

ABSTRACT. A new notion of "Measurable space frame" is introduced; the simplest but important properties are given. Further progress and more advanced results will be given in future papers in this series.

Key words : Frame, measurable, Hilbert space.

1. **Introduction**

Let H be a separable Hilbert space (i.e. one can find in H a dense countable system of vectors). Let $\{e_j\}_{j=1}^N$, where $N \in \mathbb{N} \cup \{+\infty\}$, be a sequence of orthonormal elements in H. Then, for any element $e \in H$, we have

$$\sum_{j=1}^{N} |\langle e, e_j \rangle|^2 \leq ||e||^2. \tag{1.1}$$

This is Bessel's inequality. For *complete* orthonormal sequences $\{e_j\}_{j=1}^N$ (i.e, for orthonormal bases in H) one has a stronger property of the systems:

$$\sum_{j=1}^{N} |\langle e, e_j \rangle|^2 = ||e||^2 \quad \text{for every} \quad e \in H. \tag{1.2}$$

This is Parseval's equality.

The relations (1.1) and (1.2) are, surely, most of the main ones in the classical theory of Hilbert spaces. But we can go further, considering countable or uncountable infinite (orthogonal or not) families in Hilbert spaces, even not necessarily separable, and trying to define families for which somethings like

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(1.1) and (1.2) are fulfilled. In this way, to be short, we can get naturally an "overfilled" (in most cases, nonorthogonal) families which are very similar to Hilbert bases in separable (or nonseparable) spaces — in the sense that they may have the main properties almost like the properties of the sequences in (1.1) and (1.2). If such "overfilled" uncountable families are considered in separable H, then clearly their elements do not give us usual Hilbert bases since each such family is uncountable and so is linear dependent.

Again to be short, let us say that we are on the way to obtain a new class of orthogonal families, not bases but showing themselves like bases and, so, with properties which can be very useful in Hilbert space theory. These are *frames.*

More precisely, let $\{e_\alpha\}_{\alpha\in\mathbb{A}}$ be a family in H (we suppose usually that all the sets under consideraton are not empty, so is $\mathbb{A}$). One says that the family $\{e_\alpha\}_{\alpha\in\mathbb{A}}$ is a *frame* if there exist two constants A and B, where $0 < A \leq B < \infty$, so that for each $e \in H$ the following relations are fulfilled:

$$A\,||e||^2 \leq \sum_{\alpha\in\mathbb{A}} |\langle e, e_\alpha\rangle|^2 \leq B\,||e||^2 \tag{1.3}$$

(see [1, 2] for nice inroductions to the Frame theory).

If so, then A is a *lower (frame–)bound* and B is an *upper (frame–)bound* for this frame. If a family $E := \{e_\alpha\}_{\alpha\in\mathbb{A}}$ satisfies a condition analogues to (1.1), then we say about *Bessel family* and *Bessel frame*, if it is a frame. If the family $\{e_\alpha\}_{\alpha\in\mathbb{A}}$ satisfies a condition analogues to (1.2), then we say about *Parseval family* and in this case $\{e_\alpha\}_{\alpha\in\mathbb{A}}$ is necessarily a frame, so we get *Parseval frame.* More precisely, E is said to be a *Bessel family* (respectively, a *Bessel sequence*) if there exists $B > 0$ such that for every $e \in H$

$$\sum_{\alpha\in\mathbb{A}} |\langle e, e_\alpha\rangle|^2 \leq B\,||e||^2$$

(respectively, and the index set $\mathbb{A}$ is countable). If E is a frame, then it is said to be a *Parseval frame* if in (1.3) $A = B = 1$. More general, if $A = B > 0$ in (1.3) then the frame E is called a *tight frame.*
Two more definitions: a frame E is *uniform,* if $||e_\alpha|| = ||e_\beta||$ for all $\alpha, \beta \in \mathbb{A}$; a frame E is *exact,* if the family $E \setminus \{e_\gamma\}$ is already not frame whatever $\gamma \in \mathbb{A}$ to be.

Frame theory (cf. [1, 3]) has been developed remarkably during the last about 30 years. We would not like to consider even first (but rather interesting and sometimes non-trivial) results of the theory. Almost just now we will mention some ways to generalize the notion of frame (and then we will study one of them).

In which directions one can generalize Frame notion?

$\mathbb{A}$) One way is to replace each e_α in (1.3) by a collection of vectors $\{e_{\alpha,j}\}_j$. How to do this? The scalar products in (1.3) are, roughly speaking, (ortho)projections from H onto the corresponding one dimensional spaces. So, if we want to change the situation in the desiered direction, it is natural to consider, for each α, a subspace E_α instead of one dimensional subspace span $\{e_\alpha\}$ and to take the orthogonal projecton π_{E_α} instead of the "projection" $\langle \cdot, e_\alpha \rangle$. This is the way P. G. Casazza and G. Kutyniok [2] went. Here is their corresponding definition.

Let I be some index set, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. A family of closed subspaces $\{W_i\}_{i\in I}$ of a Hilbert space H is a *frame of subspaces with respect to* $\{v_i\}_{i\in I}$ *for* H, if there exist constants $0 < C \leq D < \infty$ such that

$$C\,||f||^2 \leq \sum_{i\in I} v_i^2 ||\pi_{W_i}(f)||^2 \leq D\,||f||^2 \quad \text{for all} \quad f \in H.$$

The notions of bounds, upper and lower bounds, tight, Parseval etc. frames are defined by a natural way.

$\mathbb{B}$) Another way for a generalization. We just change the sums in (1.3) by integrals, what means that we just replace the index set $\mathbb{A}$ by a positive measure space, e.g. (Ω, Σ, μ). At the moment, we do not discuss this possibility, since essentially just now we will give the exact definitions even for more general situation.

2. "Integral" definition of frame

We introduce a new notion of "Measurable space frame" and begin the investigation with its simplest properties.

Let us look at the condition (1.3) in the definition of frame from another point of view. Put $\Omega := \mathbb{A}$ and let μ_c be the "counting measure" on $\mathbb{A}$, i.e. μ_c is the measure on σ-algebra of all subsets of the set $\mathbb{A}$ with the property that for each point $\alpha \in \mathbb{A}$ one has $\mu_c(\{\alpha\}) = 1$. Note that for each non-negative function φ on $\mathbb{A}$ the following equality holds:

$$\sum_{\alpha\in\mathbb{A}} \varphi(\alpha) = \int_{\mathbb{A}} \varphi(\alpha)\, d\mu_c(\alpha) \equiv \int_{\Omega} \varphi\, d\mu_c.$$

Thus, the relations (1.3) can be rewritten in an "integral form", and we get that the family $\{e_\alpha\}_{\alpha\in\mathbb{A}}$ is a frame if there exist two constants A and B, where $0 < A \leq B < \infty$, so that for each $e \in H$ the following relations are fulfilled:

$$A\,||e||^2 \leq \int_{\mathbb{A}} |\langle e, \cdot\rangle|^2\, d\mu_c \leq B\,||e||^2. \tag{1.3a}$$

3. Measurable space frame

Combining this "integral idea" with ideas of P. G. Casazza and G. Kutyniok [2], we come to a natural generalization in both directions (I & II) of the notion of frames as well as the notion of frame of subspaces.

Bellow, when considering "any–valued" functions on a set with a (non-negative) measure, *we will always suppose that the functions are measurable;* "positive" will mean "positive a.e." etc. For example, considering a composition of type

$$||g(\cdot)|| : \quad \text{SET (with a measure)} \xrightarrow{g}$$
$$\{\text{collection of linear subspaces of a Hilbert space}\} \xrightarrow{||\cdot||} \mathbb{R}$$

we assume that, at least, the real–valued function $||g||$ is measurable. What about the question when the desired measure conditions are fulfilled, it is a theme of other separate investigations.

For a lot of very useful information on vector integrations or nonscalar–valued mesurable functions (which will be used, in particular, in this work which is supposed to be continued in the next papers), see a nice written book by J. Diestel and J. J. Uhl [4] (this is only the author's recommendation; the reader can use also any corresponding textbook on the topic).

Definition. Let (Ω, Σ, μ) be a measure space and let φ be a positive measurable function on Ω.

1) A family $\{f_{\omega,j}\}_{\omega\in\Omega, j\in J_\omega}$ of vectors of a Hilbert space H is said to be an *integral frame* (with respect to φ) for H if there exist some constants $0 < C \le D < \infty$ for which

$$C\|f\|^2 \le \int_\Omega \varphi^2 \sum_{j\in J_\omega} |\langle f, f_{\omega,j}\rangle|^2 \, d\mu \le D\|f\|^2 \quad \forall\, f \in H. \tag{1.4}$$

More general,

2) A family $(F_\omega)_{\omega\in\Omega}$ of closed (linear) subspaces of a Hilbert space H is said to be *a measurable space frame (MSF),* for H with respect to φ, if there exist some constants $0 < C \le D < \infty$ for which

$$C\|f\|^2 \le \int_\Omega \varphi^2 \|\pi_{F_\omega}(f)\|^2 \, d\mu \le D\|f\|^2 \quad \forall\, f \in H. \tag{1.4a}$$

The constants C, D are called *frame–bounds* for MSF. The collection (F_ω) is said to be *C-exact MSF* with respect to φ, if we can take the constants C, D in (1) in such a way, that $C = D$; it is said to be a *Parseval's MSF* if $C = D = 1$; if $H = \oplus \int_\Omega F_\omega$, then this Parseval's MSF is *an orthonormal basis* for H, consisting of subspaces in MSF (orthonormal MSF–basis). Furthermore, the

MSF (with(1.4, 1.4a)) is *uniform* if $\varphi = const$. If, in (1.4, 1.4a), they have only the upper estimate, then we say about *Bessel measurable family of subspaces with respect to* φ with the *Bessel upper bound* D.

Theorem 1. Let $\varphi(\omega) > 0$ for every $\omega \in \Omega$ and let $\{f_{\omega\, j}\}_{j \in J_\omega}$ be a frame in H with upper bounds B_ω (upper) and A_ω (lower). For all $\omega \in \Omega$, let us put $F_\omega = \overline{span}_{j\in J_\omega} \{f_{\omega\, j}\}$. Take, in each of the subspace F_ω an orthonormal basis $\{e_{\omega\, j}\}_{j\in J_\omega}$. Suppose that $0 < A = \inf_{\omega\in\Omega} A_\omega \leq B = \sup_{\omega\in\Omega} B_\omega < \infty$. The following conditions are equivalent:

(1) $\{\varphi(\omega)\, f_{\omega\, j}\}_{\omega\in\Omega\, j\in J_\omega}$ is an integral frame for H;

(2) $\{\varphi(\omega)\, e_{\omega\, j}\}_{\omega\in\Omega\, j\in J_\omega}$ is an integral frame for H;

(3) $\{F_\omega\}_{\omega\in\Omega}$ is a MSF with respect to φ for H.

Proof. Since, for every $\omega \in \Omega$, $\{f_{\omega\, j}\}_{j\in J_\omega}$ is a frame for H, with bounds A_ω, B_ω, then we get

$$A \int_\Omega \varphi^2 \|\pi_{F_\omega}(f)\|^2 \leq \int_\Omega A_\omega \varphi^2 \|\pi_{F_\omega}(f)\|^2 \leq \int_\Omega \sum_{j\in J_\omega} |\langle \pi_{F_\omega}(f), \varphi\, f_{\omega j}\rangle|^2 \leq \int_\Omega B_\omega \varphi^2 \|\pi_{F_\omega}(f)\|^2 \leq B \int_\Omega \varphi^2 \|\pi_{F_\omega}(f)\|^2.$$

Note that

$$\int_\Omega \sum_{j\in J_\omega} |\langle \pi_{F_\omega}(f), \varphi\, f_{\omega j}\rangle|^2 = \int_\Omega \sum_{j\in J_\omega} | < f, \varphi(\omega) f_{\omega j} > |^2.$$

Therefore, if $\{\varphi(\omega)\, f_{\omega\, j}\}_{\omega\in\Omega\, j\in J_\omega}$ is a frame for H with bounds C, D, then $\{F_\omega\}$ is a MSF for H with bounds C/B and D/A. Moreover, if $\{F_\omega\}$ is a MSF with bounds C, D, then it follows from above computations that $\{\varphi(\omega)\, f_{\omega\, j}\}_{\omega\in\Omega\, j\in J_\omega}$ is an integral frame for H with bounds AC, BD. Thus, (1) $\iff$ (3).

To prove the equivalence (2) $\iff$ (3), it is enough to note (in the above computations) that

$$\varphi^2(\omega)\, \|\pi_{F_\omega}(f)\|^2 = \varphi^2(\omega)\, \| \sum_{j\in J_\omega} < f, e_{\omega j} > e_{\omega j} \| = \sum_{j\in J_\omega} | < f, \varphi(\omega) e_{\omega j} > |^2.$$

This finish the proof.

One of the main notions in analytic theories (mathematical analysis, functional analysis, operator ideals, operator algebras etc.) is the notion of completeness.

Recall that a family of subspaces $\{F_\omega\}$ is complete in a Hilbert space H if

$$\overline{span}_{\omega\in\Omega} \{F_\omega\} = H.$$

The following useful fact takes a place.

Proposition 2. Let $\{F_\omega\}$ be a family of subspaces in H, $\varphi \in L^2$. The corresponding MSF (if MSF) is complete in H.

Proof. Suppose that the family $\{F_\omega\}$ is not complete. Take a nonzero element $f \in H$ such that $f \perp \overline{span}_{\omega\in\Omega}\{F_\omega\}$. Then $\int_\Omega \varphi^2 \|\pi_{F_\omega} f\|^2 = 0$ (because of all $\pi_{F_\omega} f$ are zero). Hence, our family is not a MSF (see (1.4)).

A simple completeness criteria (cf. [1]):

Lemma 3. Let $\{F_\omega\}$ be a family of subspaces in H and let, for each $\omega \in \Omega$, $\{e_{\omega j}\}_{j\in J_\omega}$ be an orthonormal basis in F_ω. The following assertions are equivalent:

(1) $\{F_\omega\}$ is complete in H;

(2) $\{e_{\omega j}\}_{\omega\in\Omega, j\in J_\omega}$ is complete.

Let us mention without a proof (which will appear in the next part of the work) one interesting fact which is an analogue of a result of Pete Casazza (Proposition 4 is not very trivial assersion).

Proposition 4. If we throw out of some MSF a subspace then we will get either an MSF again with the same function φ or a noncomplete family of subspaces.

Finally, a hereditary property of MSF's.

Proposition 5. Let $\{F_\omega\}$ be a MSF for H with a function φ and with bounds C, D. If E is a subspace of a Hilbert space H, then $\{F_\omega \cap E\}$ is an MSF for E with the same generating function and with the same bounds.

Proof. For $f \in E$, we have:

$$\int_\Omega \varphi^2 \|\pi_{F_\omega}(f)\| = \int_\Omega \varphi^2 \|\pi_{F_\omega \cap V}(f)\|^2.$$

In next parts (next papers) of our investigations we will study Parseval MSFs, uniform MSFs, Bessel measurable family of subspaces etc., coming later to some applications in wavelets theory.

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