

METHODS OF CONSTRUCTING A KNIGHT'S TOUR ON RECTANGULAR AND DIAMOND BOARDS

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ABSTRACT. In this paper, we propose methods of constructing a closed knight's tour on a rectangular board of size $m \times n$, for $m, n \geq 6$ and mn is even, as well as on a diamond board.

Key words : Knight tour, rectangular board, diamond board.

1. Introduction

A *knight's tour* is a series of moves made by knight visiting every squares of a chessboard exactly once. The *knight's tour problem* is the problem of constructing such a tour on a given chessboard. A knight tour is called *closed* if the last square visited is also reachable from the first square by a single knight's move. Otherwise, it is called *open*.

There are several algorithms proposed to construct a knight's tour on a $m \times n$ chessboard. Paul Cull and De Curtins [1] proposed an algorithm to construct an open knight's tour on $m \times n$ board with $m, n \geq 5$. Their algorithm works by dividing the board into 2 parts of the same size. In the first part, a closed knight's tour is formed and in the second part an open one is constructed. Then, these two tours are merged into one open knight's tour. In [2], Ian

Parberry proposed another algorithm to construct a closed knight's tour on board of sizes $n \times n$, or $n \times (n + 2)$ for even $n \geq 6$, and for board of size $n \times (n + 1)$ for $n \geq 6$. The algorithm works by dividing the board into 4 quadrants, generate a closed knight's tour for each quadrant, and finally remove 4 edges and add 4 new edges at the inside corners combine four smaller closed knight's tours into a complete closed knight's tour for the board. This kind of algorithm is called *quadrisection*. These two algorithms work for a $m \times n$ board. Note that it is clear by parity argument that for mn is odd there is no closed knight's tour on the $m \times n$ board.

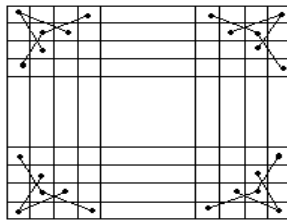


FIGURE 1. The structured knight's tour.

In this paper, we present a new method for constructing a closed knight's tour on the $m \times n$ board, as well as on a diamond board. A knight's tour is called *structured* if it includes the knight's moves shown in Figure 1. All algorithms proposed in this paper are based on structured knight tours.

In general, for obtaining a complete knight's tour for larger board we will merge two or more structured knight's tours from smaller boards by removing moves *A* and *B* (in Figure 2) and adding moves *C* and *D* (as in Figure 3). This merge operation is called a *horizontal merge operation*. A *vertical merge operation* can be defined similarly.

2. On a $m \times n$ board

In this section, we shall give a new method of constructing a closed structured knight's tour on a $m \times n$ board, for $m, n \geq 6$ and mn is even. The method is given in Theorem 1. The following property of integer numbers will be useful in the construction of such a closed tour.

Lemma 1. *Let $n \in \mathbb{Z}$ and $n \geq 12$. Then n can be expressed as $n = 6k_1 + 7k_2 + 8k_3 + 9k_4$, for some $k_1, k_2, k_3, k_4 \in \mathbb{Z}$. Moreover, if n even, then $n = 6k_1 + 8k_2$, for some $k_1, k_2 \in \mathbb{Z}$.*

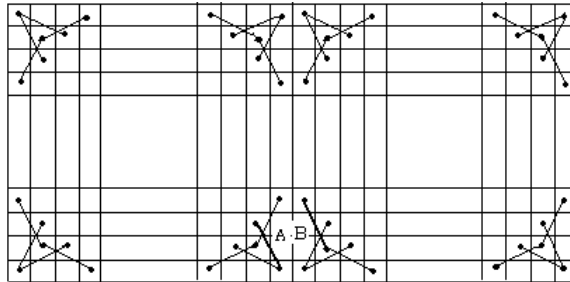


FIGURE 2. The removed edges A and B.

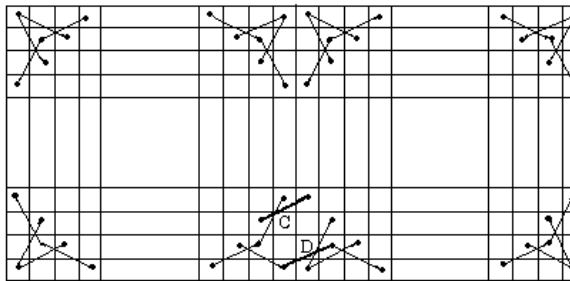


FIGURE 3. The added edges C and D.

Proof. Let $a = (n \bmod 6)$. By considering the values of a and since $n \geq 12$, then n can be expressed as: $n = 6k, 7 + 6k, 8 + 6k, 9 + 6k, 8 + 8 + 6k, 8 + 9 + 6k$ if $a = 0, 1, 2, 3, 4, 5$ respectively. Moreover, if n is even then $a = 0, 2$, or 4 . Therefore, $n = 6k, 8 + 6k$, or $8 + 8 + 6k$ for some $k \in \mathbb{Z}$. \square

To construct a closed knight's tour on a $m \times n$ board in general, we will compose structured knight's tours on the $6 \times n, 8 \times n$, and/or $10 \times n$ boards. The following lemmas give us a structured knight's tour on these latest sizes.

Lemma 2. *For any $n \geq 6$, there exists a closed structured knight's tour on a $6 \times n$ board.*

Proof. For $6 \leq n \leq 11$, the tours are given in Figures 4 and 5. Now, suppose $n \geq 12$. By Lemma 1 we know that $n = 6k_1 + 7k_2 + 8k_3 + 9k_4$, for some $k_1, k_2, k_3, k_4 \in \mathbb{Z}$. Partition the $6 \times n$ board into $(6 \times 6k_1), (6 \times 7k_2), (6 \times 8k_3)$, and $(6 \times 9k_4)$ boards. In each partition we can construct a closed structured knight's tour by horizontal merge operations. Then, all obtained tours can be composed into one complete closed structured knight's tour on the $6 \times n$ board by horizontal merge operations. \square

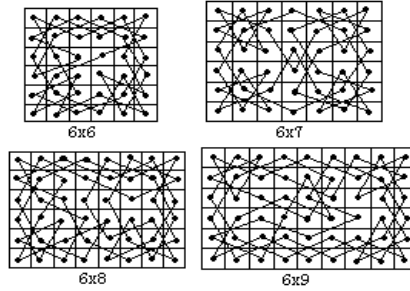


FIGURE 4. The structured knight's tour on 6×6 , 6×7 , 6×8 , and 6×9 boards.

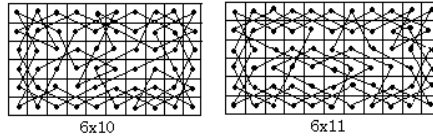


FIGURE 5. The structured knight's tour on 6×10 , and 6×11 boards.

Lemma 3. *For any $n \geq 6$, there exists a closed structured knight's tour on a $8 \times n$ board.*

Proof. For $6 \leq n \leq 11$, the tours are given in Figures 6 and 7. Since $n \geq 12$, by Lemma 1 we get $n = 6k_1 + 7k_2 + 8k_3 + 9k_4$, for some $k_1, k_2, k_3, k_4 \in \mathbb{Z}$. Partition the original $8 \times n$ board into $(8 \times 6k_1)$, $(8 \times 7k_2)$, $(8 \times 8k_3)$, and $(8 \times 9k_4)$ boards. In each partition, we can construct a closed structured knight's tour by horizontal merge operations. Compose all the obtained tours by horizontal merge operations into one larger closed structured knight's tour on the original board. \square

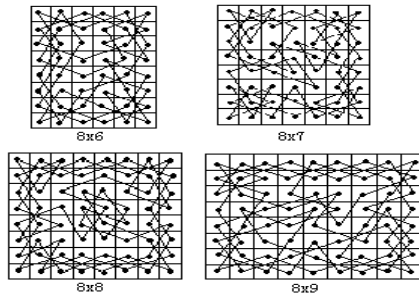


FIGURE 6. The structured knight's tour on 8×6 , 8×7 , 8×8 , and 8×9 boards.

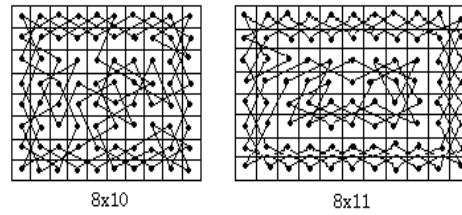


FIGURE 7. The structured knight's tour on 8×10 , and 8×11 boards.

Lemma 4. *For all $n \geq 6$, there exists a closed structured knight's tour on a $10 \times n$ board.*

Proof. For $6 \leq n \leq 11$, the tours are given in Figures 8 and 9. The construction of a closed structured knight's tour on the $10 \times n$ board can be done in a similar way of the one in Lemma 2. \square

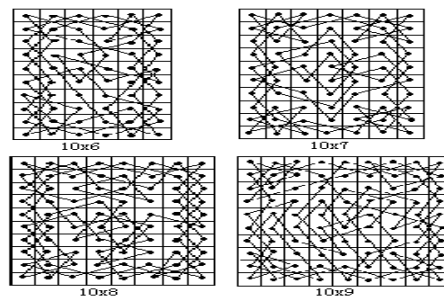


FIGURE 8. The structured knight's tour on 10×6 , 10×7 , 10×8 , and 10×9 boards.

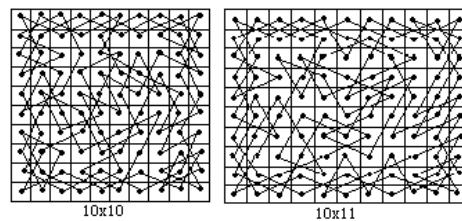


FIGURE 9. The structured knight's tour on 10×10 , and 10×11 boards.

Theorem 1. *For any $m, n \geq 6$ and mn is even, there exists a closed structured knight's tour on a $m \times n$ board.*

Proof. For $6 \leq m, n \leq 10$ and mn is even, the theorem has been proved by Lemmas 2, 3 and 4. Now, suppose $m, n \geq 12$ and mn is even. Since mn is even then either m or n is even. Without loss of generality we can assume that m is even, then by Lemma 1, $m = 6k_1 + 8k_2$, for some $k_1, k_2 \in \mathbb{Z}$. Now, partition the board of size $m \times n$ into $(6k_1 \times n)$ and $(8k_2 \times n)$ boards. By Lemma 2 and the use of the vertical merge operation, we can obtain a closed structured knight's tour on the $6k_1 \times n$ board. By using Lemma 3 and the vertical merge operation again, we may obtain a structured knight's tour on $8k_2 \times n$ board. Finally, these two obtained tours can be merged vertically to form a complete closed structured knight's tour on the $m \times n$ board. \square

3. On the diamond board

A *step board* of size n is a board (as shown in Figure 10) composed by s different size $n_i \times k_i$ boards, where $n_i, k_i \geq 6$, and $n = n_s = k_1 + k_2 + \dots + k_s$. Integers n_i and k_i are called the *length* and the *height* of the step i , respectively. A *diamond board* of size n is a board consists of four step board of size $\frac{n}{2} \times \frac{n}{2}$ and shown in Figure .

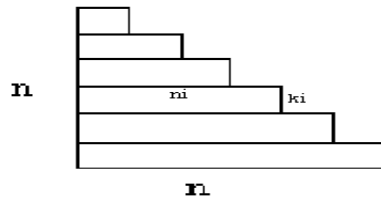


FIGURE 10. The step board n

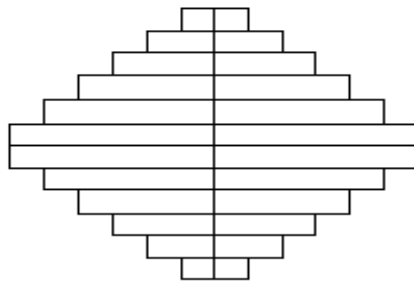


FIGURE 11. The diamond board

In this section, we propose a method of constructing a closed structured knight's tour on a diamond board of size n , with $n \geq 24$. The method is based on the construction of knight's tour on step boards.

Lemma 5. *For any integer $n \geq 12$, there exists a closed structured knight's tour on a step board of size n , with each length and height are greater than 6 and even height.*

Proof. Let the step board consist of s boards with sizes $n_i \times k_i$, where $n_i, k_i \geq 6$, and $n = n_s = k_1 + k_2 + \dots + k_s$. Since k_i is even for each i then by Lemma 2 and 3, there exists a closed structured closed knight's tour on each $k_i \times n_i$ board. By vertical merge operations we get a complete closed structured closed knight's tour on the step board. \square

Theorem 2. *For any integer n divisible by 4 and $n \geq 24$, there exists a closed structured knight's tour on diamond board of size n .*

Proof. The diamond board of size n can be divided into four quadrants. Each quadrant is exactly a step board of size $m \times m$, where $m = \frac{n}{2}$ is even. If $n \geq 24$, then $m \geq 12$. By Lemma 5, we can obtain a closed structured knight's tour on each step board of size $m \times m$. In order to construct a structured knight's tour on the original diamond board, delete edges A, B, C, D, E , and F shown in Figure 12 and add new six edges shown Figure 13. Thus, we get a closed structured knight's tour on diamond board of size n . \square

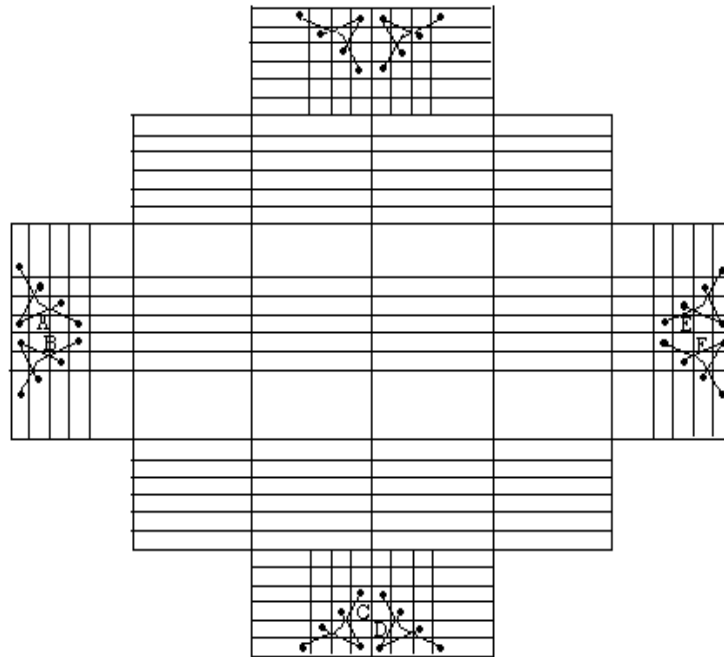


FIGURE 12. The deleted edges A,B,C,D,E,and F in a diamond board

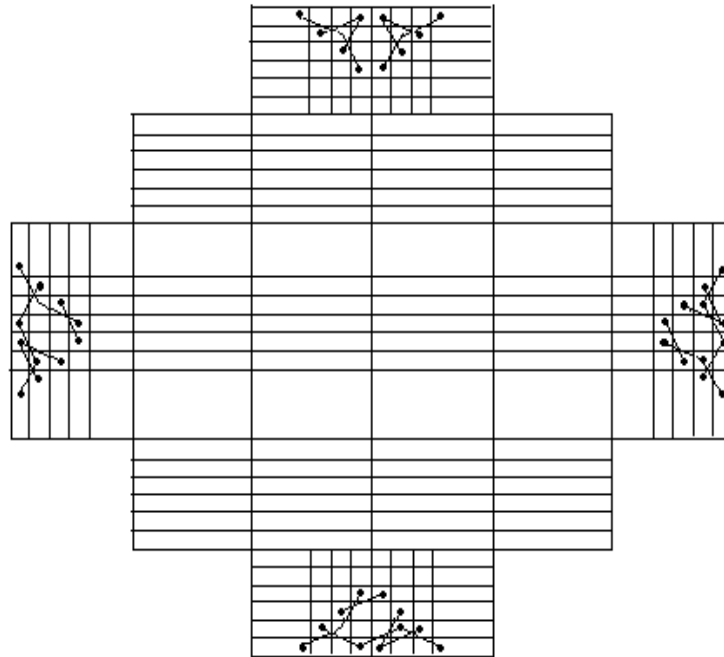


FIGURE 13. The added six edges in a diamond board

For example, Figure 14 shows a closed knight's tour on diamond board size 36.

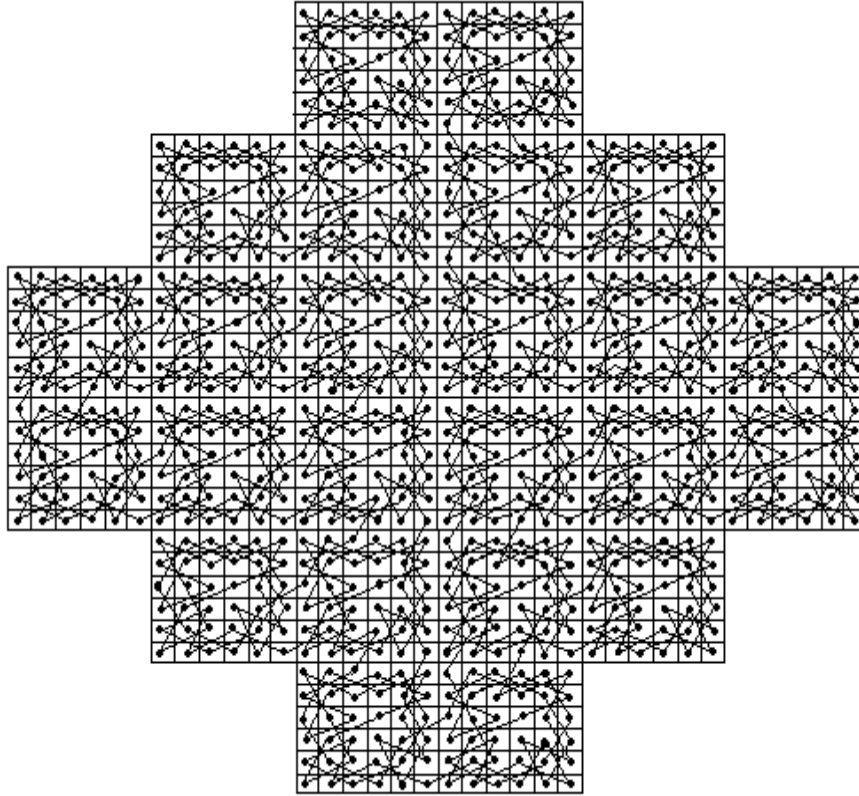


FIGURE 14. A closed structured knight's tour on a diamond board of size 36.

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