IMO 2005

First National Mathematics Olympiad Camp

1. First Round (Augest 22 - 28, 2004)

Time allowed: 3 Hours

Problem.1. Let $(1+\sqrt{2})^n = A_n + B_n\sqrt{2}$

Where $n \in N$ and $A_n, B_n \in Q$

Find

$$\lim_{n\to\infty} \frac{A_n}{B_n}$$

Problem.2. Prove that $(p+1)$ divides C_p^{2p} for all $p \in N$.

Problem.3. Let a, b, c be positive reals. First show that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

then generalize the above inequality and give a proof of it.

2. Second Round (November 30, 2004)

Time allowed: 3 Hours

Problem.1. Find all positive integers n with the following property:

There exist integers a and b such that $n^2 = a + b$ and $n^3 = a^2 + b^2$.

Problem.2. Let a, b, c be the lengths of the sides of a triangle ABC.

a) Prove that there exists a triangle whose sides have lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$.

b) Show that

$$1 \leq \frac{a+b+c}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} < 2.$$

Problem.3. Let ABC be a right angle triangle such that $\angle A = 90^\circ$ and $\angle B > \angle C$. An arbitrary point D is taken on the segment BC.
The angle bisector of $\angle ADB$ intersects AB in M and the angle bisector of $\angle ADC$ intersects AC in N. The line MN intersects BC in P. Show that $\angle MPB = \frac{1}{2}(\angle B - \angle C)$ if and only if D is the foot of the altitude from A.

3. **Final Round (December 02, 2004)**

Time allowed: 3 Hours

Problem.1. In a given triangle ABC; O is the circumcenter; D is the midpoint of the side AB and E is the centroid of the triangle ACD.

a) Show that if $AB = AC$, then the lines CD and OE are perpendicular.

b) Show that if CD and OE are perpendicular, then $AB = AC$.

Problem.2. We are given a rectangle $ABCD$ whose sides have lengths $AB = CD = a$, $BC = AD = b$ and $a < b$. A point P is moving along the perimeter of the rectangle. Find the positions of P for which the sum $PA + PB + PC + PD$ is least.

Problem.3. Let n be a positive integer which is divisible by 7. Show that the number $N = (n+1)(n+2)(n+3)(n+4)(n+5)(n+6)$ cannot be a perfect square.

Problem.4. Let a, b, c be non negative real numbers which are not greater than 1 and such that $ab + bc + ca = 1$. Show that $a^2 + b^2 + c^2 \leq 2$.

IMO 2006

Second National Mathematics Olympiad Camp

4. **Tests for NMO at different cities of Pakistan**
First Test:

4.1. **Islamabad Center:.**

Time allowed: 90 Minutes

Problem.1. Let x be a real number such that $x^4 + \frac{1}{x^4} = 2$. Find the possible values of $x + \frac{1}{x}$.

Problem.2. How many 4-digit numbers are there which begin with 1 and have exactly two identical digits? (Examples are $1557, 1030, 1321$.)

Problem.3. Let P be a point in the interior of the triangle ABC. The reflections of P across the midpoints of the sides BC, CA, AB, are $P_A, P_B, and\ P_C$ respectively. Prove that the lines $AP_A, BP_B, and\ CP_C$ are concurrent.

4.2. **Karachi Center:.**

Time allowed: 90 Minutes

Problem.1. Find all positive integers n such that the number $n^8 + n^4 + 1$ is a prime.

Problem.2. Let $n \geq 2$ be an integer and let $x_1, x_2, ..., x_n \in \{-1, 1\}$ such that

$$x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_nx_1 = 0$$

Prove that n is divisible by 4.

Problem.3. Let $ABCD$ be a convex quadrilateral. Prove that there exists a circle in its interior touching all its sides if and only if $AB+CD = AD+BC$.

4.3. <u>**Lahore Center:**</u>.

Time allowed: 90 Minutes

Problem.1. Let m, n be positive integers such that $\frac{7}{10} < \frac{m}{n} < \frac{11}{15}$. Find the smallest possible value of n.

Problem.2. Denote by P the perimeter of triangle ABC. If M is a point in the interior of the triangle, prove that

$$\frac{1}{2}P < MA + MB + MC < P.$$

Problem.3. Prove that if an arithmetical progression of positive integers contains a square, then it contains infinitely many squares.

4.4. <u>**Peshawar Center:**</u>.

Time allowed: 90 Minutes

Problem.1. Let a, b, c be real numbers. Prove the inequalities:

a) $a^2 + b^2 + c^2 \geq ab + bc + ca$;

b) $a^4 + b^4 + c^4 \geq abc(a + b + c)$.

Problem.2. Prove that the product of four consecutive positive integers:

a) is divisible by 24;

b) is never a perfect square.

Problem.3. The side length of the equilateral triangle ABC equals l. The point P lies in the interior of ABC and the distances from P to the triangle's sides are $1, 2, 3$. Find the possible values of l.

4.5. **Skardu Center:.**

Time allowed: 90 Minutes

Problem.1. How many solutions has the equation $A \cup B = \{1, 2, 3, 4\}$?

(two solutions (A_1, B_1) and (A_2, B_2) are different if and only if $A_1 \neq A_2$ or $B_1 \neq B_2$; for instance, $A_1 = \{1, 3, 4\}$, $B_1 = \{1, 2, 4\}$ and $A_2 = \{1, 2, 4\}$, $B_2 = \{1, 3, 4\}$ are different solutions.)

Problem.2. Can you find a polynomial $f \in Z[X]$ such that $f(Z) \subset N$ and f takes the value -2005?

Problem.3. Compute $cos\frac{\pi}{5}$

First Training Camp(Augest 17 - 23, 2005)

First Test:

Time allowed: 2 Hours

Problem.1. Given any five integers , show that there is at least one subset of three integers Whose sum is divisible by three, e.g. $\{9, -13, 1\}$ whose sum is divisible by three.

Problem.2. Finds the value of

$$\sqrt{1 - \sqrt{\frac{17}{16} - \sqrt{1 - \sqrt{\frac{17}{16} - \sqrt{1 - ....}}}}}$$

Assuming that it is a real number.

Problem.3. The radius of the circum circle of an isosceles triangle is R and the radius of its Inscribed circle is r. Prove that the distance between the two centers is $\sqrt{R(R - 2r)}$.

Problem.4. Find all the positive integers n such that the number $2^4+2^7+2^n$ is a perfect Square.

Problem.5. Prove the following equality

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\ldots\ldots\ldots\ldots+\sqrt{1+\frac{1}{2004^2}+\frac{1}{2005^2}}=2005-\frac{1}{2005}$$

Final Test:

Time allowed: 3 Hours

Problem.1. Let a, b, c be the positive numbers such that $ab+bc+ca=1$. Show that $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\sqrt{3}+\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}$

Problem.2. A sequence $(a_n)_{n\in N}$ is given by

$$a_1=1; a_n=\frac{4n-2}{n}a_{n-1}; n\geq 2$$

Prove that all the terms of the sequence are positive integers.

Problem.3. Consider the two circles in the diagram intersecting at points A and D. Let ARC and BPD be the straight lines. DR extended meets BC at point S. Prove that PR is parallel to BC. (BC is not tangent to the smaller circle).

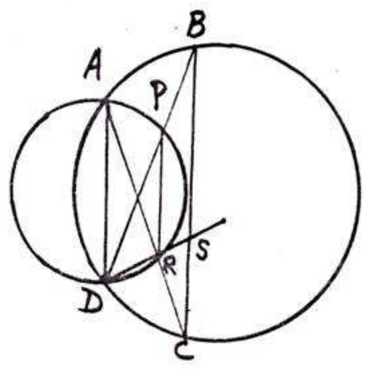

FIGURE 1. circle

First selection test for IMO (January 24, 2006)

First Test:

Time allowed: 3 Hours

Problem.1. The circle inscribed in the triangle ABC touches the sides BC, CA, AB
at points A',B', C', respectively. Prove that the lines AA', BB',

CC' are concurrent.

Problem.2. Show that the equation $x^2 + x + 2 = 20y^3$ has no integral

solutions.

Problem.3. Let $ABCD$ be a convex quadrilateral. Consider the points A'

and B' such that C is the midpoint of the line segment AA' and D is the

midpoint of the line segment BB'. Evaluate with proof, $\frac{[ABA'B']}{[ABCD]}$.

Problem.4. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{2} \leq \frac{a}{a+2b+3c} + \frac{b}{b+2c+3a} + \frac{c}{c+2a+3b} < 1.$$

Problem.5. Find, with proof, the greatest number of elements of a subset

M of the set $A = \{1, 2, , 2006\}$ having the following property: The sum of

any two distinct elements of M is not divisible by 12.

Second selection test for IMO (January 30, 2006)

Second Test:

Time allowed: 3 Hours

Problem.1. Let $ABCD$ be a convex quadrilateral, such that $BC = CD$ and

$\angle BCD = 90°$. Prove that $\frac{AB+AD}{AC} = \sqrt{2}$. if and only if $\angle BAD = 90°$.

Problem.2. a) Find two numbers $a, b \in N$ such that b is not divisible by a and, for any $k = 1, 2, ..., 10;$ $\qquad a^k$ divides b^{k+1} $\qquad$ (*)

b) What can you say if (*) is true for infinitely many k?

Problem.3. Let $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{5}$ and $x_{n+2} = \frac{x_{n+1}x_n}{5x_n - 6xn+1}, \forall n \geq 1$. Find a closed formula for x_n.

Problem.4. 2006 points are chosen inside a circle having area 1000 such that no three of them are collinear. Prove that one can choose three points which are the vertices of a triangle with area less than 0.499.

Third selection test for IMO (January 31, 2006)

__**Third Test:**__

Time allowed: 3 Hours

Problem.1. Let P be a point inside the triangle ABC and let P_a, P_b, P_c be its projections onto the sides BC, CA, AB, respectively. Prove that

$$PP_a^2 + PP_b^2 + PP_c^2 \geq \frac{4S^2}{a^2 + b^2 + c^2}.$$

where S denotes the area of triangle ABC.

Problem.2. Solve the equation $x^2 + 2x = 5y^2 + 4$ with integral x and y.

Problem.3. Let $a_\circ = a_1 = 1$ and $a_{n+1} = \frac{a_n(a_n+1)}{a_{n-1}}$, $\forall n \geq 1$. Prove that all terms of the sequence are positive integers.

Problem.4. In a class with 20 children, each of them has at least 10 friends. Prove that there exist two groups of 3 children such that every child of one group is a friend of any child of the second group.

Final selection test for IMO (April 09, 2006)

Final Test:

Time allowed: 3 Hours

Problem.1. a) Find all polynomials functions P verifying for all $x \in R$

$$P(x) + 1 = \frac{[P(x+1) + P(x-1)]}{2}$$

b) Are there non polynomial functions $f : R \to R$, which satisfy the same equation?

Problem.2. Consider a connected graph such that among its vertices there are the vertices A, B, C, D of a given square and the sum of the lengths of its edges is minimal.

Problem.3. We are given a $n \times k$ rectangle which is divided into nk unit cells distributed in n rows and k columns. How many ways can we assign either $+1$ or -1 in each cell such that the following conditions are satisfied:

a) The product of all elements of a row equals 1,

b) The product of all elements of a column equals -1 ?

Problem.4. Find all integer values of $k \geq 1$ satisfying the following:

There exists an odd number a such that all the remainders (mod 2^k)

of $a, a^2, a^3, ..., a^b$ (where $b = 2^{k-1}$) are different.

3rd National Mathematics Olympiad

November 10, 2007

JUNIORS: Grade 9 and 10

1. There are given two infinite sequences

$$(a_n)_{n\geq 1} : 1\cdot 1, 2\cdot 3, 3\cdot 5, 4\cdot 7, ...$$

$$(b_n)_{n\geq 1} : 1\cdot 3, 2\cdot 5, 3\cdot 7, 4\cdot 9, ...$$

Find general formulae for the two sequences and find how many common elements they have.

2. Can you find a polynomial f with integer coefficients such that $f(\mathbb{Z}) \subseteq \mathbb{N}$ and f takes the value -2007?

3. Consider a right circular cone with the vertex V and O the center of the base. The radius of the base is 1 and the height VO is $\sqrt{35}$. Consider also the diameter AB of the base and C, the midpoint of VA. Compute the length of the shortest path, on the surface of the cone, from B to C.

4. Given positive real numbers $a < b < c$, find a permutation x, y, z of them such that the sum

$$S = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

takes the maximal possible value.

5. In the plane there are five lines $l_1, l_2, ..., l_5$, any two of them non-parallel and any three nonconcurrent. Show that at least one of the angles between the given lines is $\leq 36°$.

Time: 3 hours.

Note: Attempt all problems. Each Problem is worth 7 points.

3rd National Mathematics Olympiad

November 10, 2007

SENIORS: Grade 11 and 12

1. If $n \in \mathbb{N}$, solve the equation $\cos^n x - \sin^n x = 1$, for $x \in \mathbb{R}$.

2. Let $a, b, c \in \mathbb{R}$, $a^2+b^2 \neq 0$ and $L = \{(x, y) \in \mathbb{Q}\times\mathbb{Q} \mid ax+by = c\}$.
 a) Can you find an example of (a, b, c) such that $L = \emptyset$?
 b) Can you find an example of (a, b, c) such that L has a unique element?
 c) Prove that if L contains two elements, then L has infinitely many.

3. Show that the set

$$A = \{\pm 1 \pm 2 \pm 3 \pm ... \pm 2006\}$$

contains an even number of elements.

4. Find a polynomial $P(X)$ of degree n , with real coefficients, satisfying

$$P(X)|P(X^2 - 2007X + 2007).$$

5. A $n\times\ n$ matrix contains elements only $+1$ or -1, such that on any row, on any column, and on the two diagonals of the matrix the product of elements is -1. What are the possible values of n $(n \in \mathbb{N})$?

Time: 3 hours.

Note: Attempt all problems. Each Problem is worth 7 points.